Linear systems and convolutional codes over von Neumann regular rings

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Abstract

We survey several research results in Control Theory, Systems Theory and Codes by focusing on the underlying structure of commutative von Neumann regular rings, which are of particular interest as a generalization of Boolean rings. Moreover, the class of commutative von Neumann regular rings is shown to be the largest class of commutative rings where usual control theory results generalize.

Linear systems

• A **linear system** Σ over a commutative ring R is a triple $(X, f, \mathcal{B})$, where X is the state-space R-module, f is an endomorphism of X and $\mathcal{B} \subseteq X$ is the (finitely generated) R-submodule of controls. If X is free of rank n, then $\Sigma = (A, B)$ where A is the associated matrix of f, and $Im(B) = \mathcal{B}$.

Figure 1. Systems: input-state-output description

$\bullet \xrightarrow{\text{inputs}}$ **Black box** *(states)* $\xrightarrow{\text{outputs}} \bullet$

Σ is **reachable** if $N_n^{\Sigma} = R^n$ where N_n^{Σ} is the **set of reachable states of** Σ; that is, the image of $[B|AB|\cdots|A^{n-1}B]$.

• The **problem of cyclic accesibility of reachable states:** Find matrix K and vector u with $N_n^{(A+BK,Bu)} = N_n^{(A,B)}$, see [13]. If Σ is reachable, this is the **feedback cyclization problem** [1].

• The **invariant factor assignment problem** called (A, B, M): Find F such that the pencil $sI - (A+BF)$ has $M = \mathrm{diag}\{\phi_1(s), \ldots, \phi_n(s)\}$ as Smith normal form where $\phi_1(s), \ldots, \phi_n(s)$ in $R[s]$, are n polynomials such that $\phi_i(s)|\phi_{i+1}(s)$ for $i = 1, \ldots, n-1$, see [5].

• Let $\Sigma = (X, f, \mathcal{B})$ and $\Sigma' = (X', f', \mathcal{B}')$ be linear systems over R. Σ is **feedback isomorphic** with Σ' iff there exists an isomorphism $\phi : X \to X'$ such that: $\phi(\mathcal{B}) = \mathcal{B}'$ and $\mathrm{Im}(\phi f - f'\phi) \subset \mathcal{B}'$ (see [4]).

$$\begin{array}{ccccc} \mathcal{B} & \hookrightarrow & X & \xrightarrow{f} & X \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ \mathcal{B}' & \hookrightarrow & X' & \xrightarrow{f'} & X' \end{array}$$

• A linear system Σ over R is **locally Brunovsky** iff the system Σ is reachable and its localizations $\Sigma_{\mathfrak{p}} = (X_{\mathfrak{p}}, f_{\mathfrak{p}}, \mathcal{B}_{\mathfrak{p}})$ are feedback equivalent to a Brunovsky canonical form (see [3]) for all prime ideals $\mathfrak{p}$ of R (see [6]).

If R is a commutative von Neumann regular ring, then the number of classes of locally Brunovsky linear systems with state space $X \simeq R^n$ is $(p_{\mathbb{N}}(n))^{\sharp \mathrm{Spec}(R)}$ (see [8]).

For an effective calculation of canonical forms for systems over von Neumann regular rings, see [14].

Characterization of von Neumann regular rings

A **von Neumann regular ring** R corresponds with Bourbaki's **absolutely flat rings** [2].

Let R be a commutative ring with 1. The following statements are equivalent:

(i) R is a von Neumann regular ring.

(ii) For any system (A, B), the problem of cyclic accesibillity of reachable states is solvable, see [13].

(iii) An invariant factor assignment problem (A, B, M) is solvable over R if and only if $(A_{\mathfrak{m}}, B_{\mathfrak{m}}, M_{\mathfrak{m}})$ is solvable over $R/\mathfrak{m}$, for all maximal ideals $\mathfrak{m}$, see [5].

(iv) Two systems Σ, Σ' are feedback equivalent over R if and only if for every maximal ideal $\mathfrak{m}$, $\Sigma(\mathfrak{m})$ and $\Sigma'(\mathfrak{m})$ are feedback equivalent over $R/\mathfrak{m}$ where $\Sigma(\mathfrak{m})$ denotes the extension of Σ by change of scalars under the natural quotient map $R \to R/\mathfrak{m}$, see [6].

(v) A system Σ is reachable if and only if it is locally of Brunovsky type, see [8].

Convolutional codes

A submodule $\mathfrak{C} \subset R[z]^n$ such that $R[z]^n/\mathfrak{C}$ is R-flat and $\mathrm{rk}(\mathcal{C})(\mathfrak{p}) = k$ for any prime ideal can be considered as a rate (n, k) **family of convolutional codes** of degree δ over R parametrized by $\mathrm{Spec}(R)$.

A first order representation for $\mathfrak{C}$ is a triple of matrices (K, L, M) satisfying $\mathfrak{C} = \mathrm{Ker}(f_1|f_2)$ where $f_2 = zK + L$ and $f_1 = M$. Moreover, if the following conditions are verified:

1. the matrix $K : R^{\delta} \to R^{\delta+n-k}$ is injective with flat cokernel
2. the matrix $(K, M) : R^{\delta+n} \to R^{\delta+n-k}$ is surjective
3. the matrix $(zK + L, M) : R[z]^{\delta+n} \to R[z]^{\delta+n-k}$ is surjective,

then this representation is called minimal.

Over a commutative von Neumann regular ring R, every free submodule of $R[z]^n$ is a family of convolutional codes. If R is also Noetherian, then there exists a (unique) minimal first order representation for any (n, k) family of convolutional codes, $\mathfrak{C}$, of degree δ.

Moreover, there exist matrices (A, B, C, D) over R that are a **reachable I/S/O (input/state/output) representation** for each $\mathfrak{C}$. See [7], which generalizes to rings what was done in [12] for fields.

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References

[1] J.W. Brewer, J.W. Bunce, F.S. Van Vleck, Linear systems over commutative rings, Marcel Dekker, 1986.

[2] N. Bourbaki, Éléments de mathématique, Commutative Algebra, Ch. 1-7, Springer-Verlag, 1989.

[3] P.A. Brunovsky, A classification of linear controllable systems, *Kybernetika* 3 (1970), 173–187.

[4] M. V. Carriegos, Enumeration of classes of linear systems via equations and via partitions in an ordered abelian monoid, *Linear Algebra Appl.* 438 (2013), 1132–1148.

[5] M.V. Carriegos, J.A. Hermida-Alonso, A. Sáez-Schwedt, T. Sánchez-Giralda, Rosenbrock's theorem for systems over von Neumann regular rings. *Linear Algebra Appl.* 482 (2015), 122–130.

[6] M.V. Carriegos, J.A. Hermida-Alonso, T. Sánchez-Giralda, The pointwise feedback relation for linear dynamical systems. *Linear Algebra Appl.* 279 (1998), 119–134.

[7] M.V. Carriegos, N. DeCastro-García, A.L. Muñoz Castañeda, Linear representations of convolutional codes over rings, preprint in arXiv: 1609.05043v1 (2016), 1-17.

[8] N. DeCastro-García, M.V. Carriegos, A.L. Muñoz Castañeda, A characterization of von Neumann regular rings in terms of linear systems, *Linear Algebra Appl.* 494 (2016), 236–244.

[9] K.R. Goodearl, Von Neumann Regular Rings, 2nd ed., Robert E. Krieger Publishing Co. Inc., 1991.

[10] J.A. Hermida-Alonso, M.P. Pérez, T. Sánchez-Giralda, Brunovsky's canonical form for linear dynamical systems over commutative rings, *Linear Algebra Appl.* 233 (1996), 131–147.

[11] R.E. Kalman, Kronecker invariants and feedback, in: Ordinary Differential Equations, Academic Press, 1972, pp. 459–471.

[12] J. Rosenthal, J. M. Schumacher, E. V. York, On behaviors and convolutional codes, *IEEE Trans. Inform. Theory* 42 (1996), 1881–1891.

[13] A. Sáez-Schwedt, Cyclic accessibility of reachable states characterizes von Neumann regular rings, *Linear Algebra Appl.* 433 (2010), 1187–1193.

[14] A. Sáez-Schwedt, W. Schmale, Feedback classification of linear systems over von Neumann regular rings, *Linear Algebra Appl.* 438 (2013), 1852–1862.

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