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Modeling Hierarchical Systems via Nested Generalized Disjunctive Programming

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Third Sevilla MINLP Workshop
Instituto Matemáticas de la Universidad de Sevilla
March 30-31, 2023

3SMINLP



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Prof. Egon Balas *(1922-2019)*

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Goal: Extend Disjunctive Programming Theory for Polyhedra to Nonlinear Convex/Nonconvex Sets

DP: Linear programming with disjunctions

Balas (1974, 1979, 1985, 1988)





2018

Generalized Disjunctive Programming *(GDP)*

Raman and Grossmann (1994) *(Extension Balas, 1979)*

$$\min \; Z = \sum_k c_k + f(x) \qquad \text{Objective Function}$$

$$s.t. \quad r(x) \le 0 \qquad \text{Common Constraints}$$

OR operator $\longrightarrow$

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \le 0 \\ c_k = \gamma_{jk} \end{bmatrix}, k \in K$$

Disjunction

Constraints

Fixed Charges

$$\Omega(Y) = true \qquad \text{Logic Propositions}$$

$$x \in R^n, c_k \in R^1 \qquad \text{Continuous Variables}$$

$$Y_{jk} \in \{ true, false \} \qquad \text{Boolean Variables}$$

a) Provides a *"high level"* modeling representation

b) Can be used to derive MI(N)LP models *(algebraic constraints)*

Goal: introduce extension to Nested GDP

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GDP is a higher level of representation for MILP/MINLP

Optimization problem with algebraic expressions, disjunctions & logic propositions

General form of GDP[1]

$$\min z = f(x) \qquad \textit{Objective Function}$$

$$s.t.\ g(x) \leq 0 \qquad \textit{Global Constraints}$$

$$\bigvee_{i \in D_k} \begin{bmatrix} Y_{ki} \\ r_{ki}(x) \leq 0 \end{bmatrix} \quad k \in K \qquad \textit{Disjunctions}$$

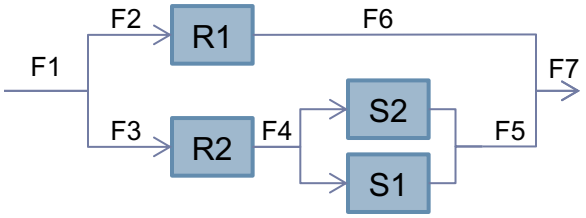
$$\veebar_{i \in D_k} Y_{ki} \qquad k \in K$$

$$\Omega(Y) = True \qquad \textit{Logic Propositions}$$

$$x \in \mathbb{R}^n$$

$$Y_{ki} \in \{True, False\} \qquad k \in K, i \in D_k$$

Illustration – Process network



$$\max z = P_7F_7 - P_1F_1 - c_R - c_S \qquad \textit{Objective function}$$

$$F_1 = F_2 + F_3$$
$$F_7 = F_5 + F_6 \qquad \textit{Global Constraints}$$

$$\begin{bmatrix} Y_{R1} \\ F_6 = \beta_{R1}F_2 \\ F_3 = F_4 = 0 \\ c_R = \gamma_{r1} \end{bmatrix} \vee \begin{bmatrix} Y_{R2} \\ F_6 = F_2 = 0 \\ F_4 = \beta_{R2}F_3 \\ c_R = \gamma_{r2} \end{bmatrix}$$

$$\begin{bmatrix} Y_{S1} \\ F_5 = \beta_{S1}F_4 \\ c_S = \gamma_{s2} \end{bmatrix} \vee \begin{bmatrix} Y_{S2} \\ F_5 = \beta_{S2}F_4 \\ c_S = \gamma_{s2} \end{bmatrix} \vee \begin{bmatrix} Y_{S_NO} \\ F_5 = 0 \\ c_S = 0 \end{bmatrix} \qquad \textit{Disjunctions}$$

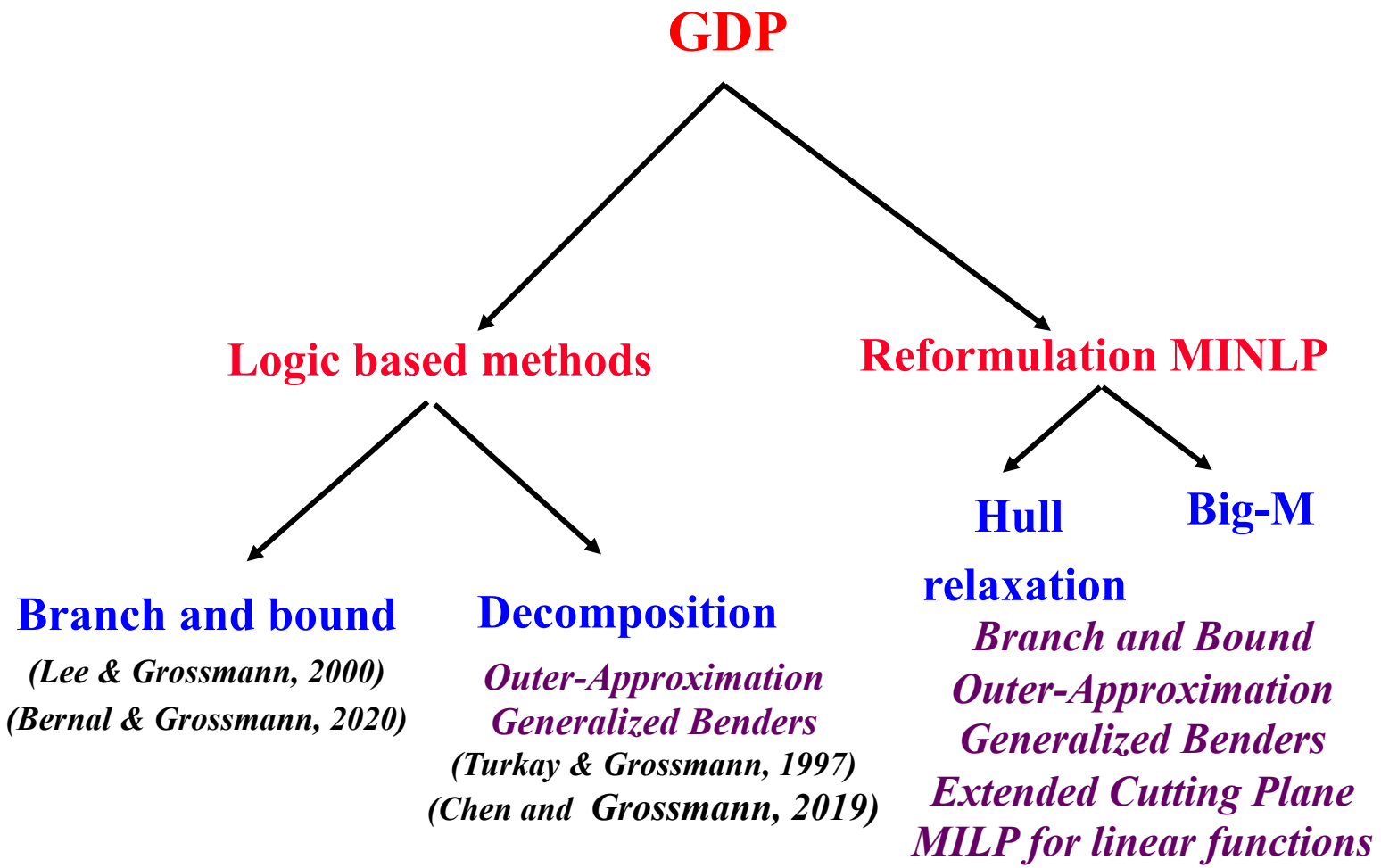
$$Y_{R1} \veebar Y_{R2}$$

$$Y_{S1} \veebar Y_{S2} \veebar Y_{S_NO}$$

$$Y_{R1} \Leftrightarrow Y_{S_NO} \qquad \textit{Logic}$$

1. Raman R. and Grossmann I.E., "Modelling and Computational Techniques for Logic-Based Integer Programming", Computers and Chemical Engineering, 18, 563, 1994.

Methods Generalized Disjunctive Programming



Big-M MINLP (BM)

- **MINLP reformulation of GDP**

$$\min Z = \sum_{k\in K}\sum_{j\in J_k} \gamma_{jk}\lambda_{jk} + f(x)$$

$$s.t. \quad r(x) \le 0$$

$$g_{jk}(x) \le M_{jk}(1-\lambda_{jk}) \quad ,j \in J_k, k \in K$$

$$\sum_{j\in J_k} \lambda_{jk} = 1, \quad k \in K$$

$$A\lambda \le a$$

$$x \ge 0, \lambda_{jk} \in \{0,1\}$$

Big-M Parameter

Logic constraints

Williams (1990)

NLP Relaxation $0 \le \lambda_{jk} \le 1$ **=> Lower bound to optimum of GDP**

Hull Relaxation Problem (HRP)

(Lee, Grossmann, 2000)

HR:

$$\min Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x)$$

$$s.t. \quad r(x) \le 0$$

Disaggregated variables

$$x = \sum_{j \in J_k} \nu^{jk}, \; k \in K$$

$$0 \le \nu^{jk} \le \lambda_{jk} U_{jk}, \; j \in J_k, \; k \in K$$

$$\sum_{j \in J_k} \lambda_{jk} = 1, \; k \in K$$

Convex Hull each disjunction

Perspective function →

$$\lambda_{jk} g_{jk}(\nu^{jk} / \lambda_{jk}) \le 0, \quad j \in J_k, \; k \in K$$

$$A\lambda \le a$$

Logic constraints

$$x, \nu^{jk} \ge 0, \; 0 \le \lambda_{jk} \le 1, \; j \in J_k, \; k \in K$$

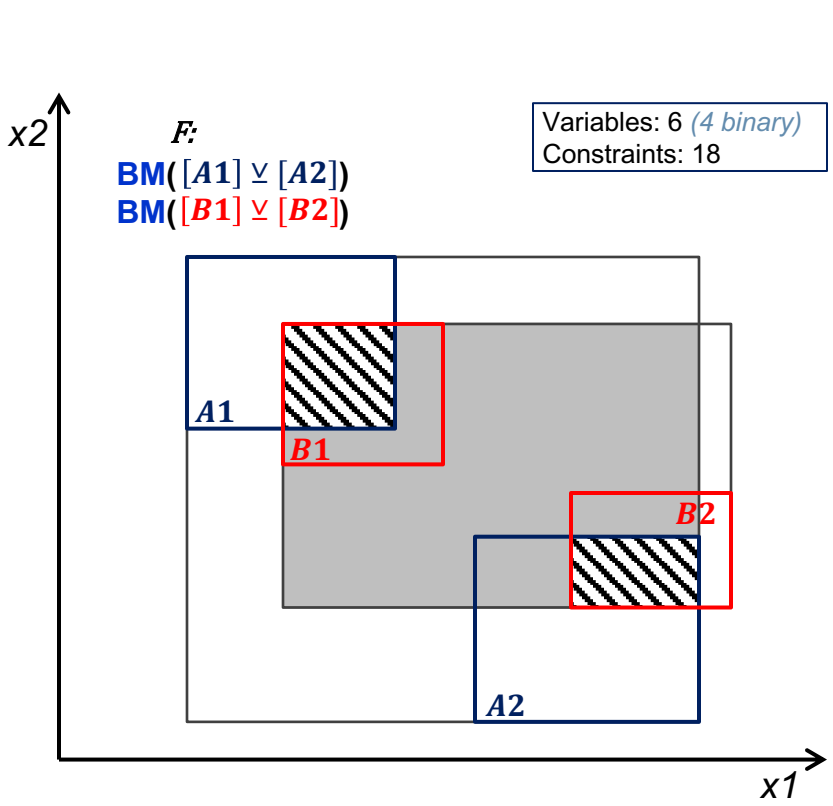
- **Property**: *The NLP (HRP) yields a lower bound to optimum of (GDP).*

MINLP reformulation: set $\lambda_{jk} = 0,1$

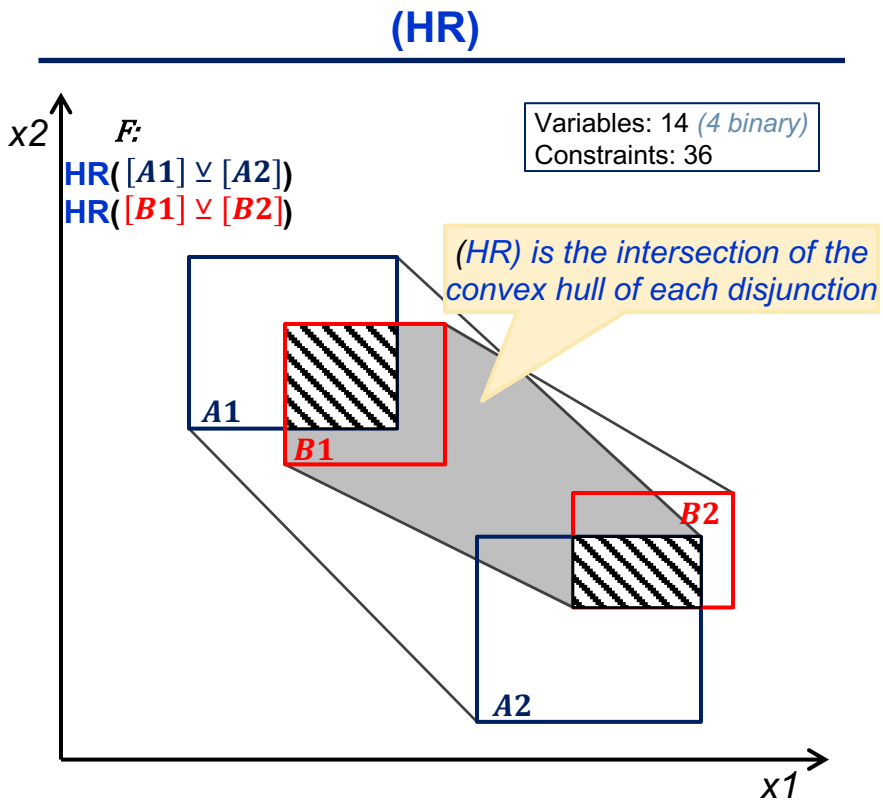
Hull relaxation: intersection of convex hull of each disjunction

(HR) provides a tighter relaxation than (BM)

- Theorem: *The relaxation of (HR) yields a lower bound that is greater than or equal to the lower bound that is obtained from the relaxation of problem (BM)* **Grossmann, Lee (2003)**



(HR)



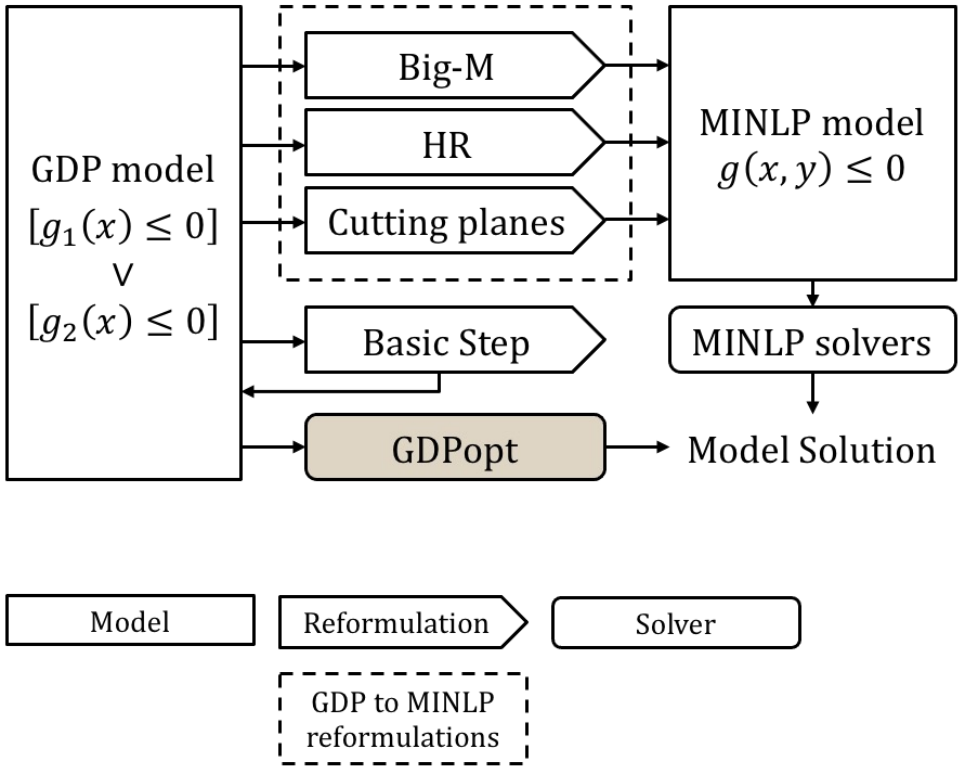
Tradeoff: (BM) has a smaller problem size while (HR) has a tighter continuous relaxation

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Pyomo.GDP flexible solution strategies

Chen, Q., E.S. Johnson, D.E. Bernal, R. Valentin, S.Kale, J. Bates, J. D. Siirola and I.E. Grossmann (2020)

- Vision: **write** model **once**, **try various** solution strategies



- GDPopt solver
  - **LOA**: Logic-based outer approximation
  - **GLOA**: Global logic-based outer approximation
    - MC++ interface – Multivariate McCormick envelope computation
    - Disjunctive range reduction using FBBT or OBBT
  - **LBB**: Nonlinear disjunctive branch and bound
    - Z3 satisfiability solver interface – infeasible node screening

PYOMO GDP

https://github.com/Pyomo/pyomo

https://github.com/IDAES/idaes-pse

https://github.com/grossmann-group/gdplib/tree/master/gdplib

IDAES Institute for the Design of Advanced Energy Systems

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Nested GDP (NGDP): Hierarchical Logic

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$

$$\bigvee_{j \in J_i} \begin{bmatrix} Y_{ij} \\ g_{ij}(x) \le 0 \end{bmatrix} \qquad \forall i \in I$$

$$\Omega(Y) = True$$

$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y_{ij} \in \{True, False\} \qquad \forall i \in I, j \in J_i$$

For problems with a hierarchical structure, there are lower-level decisions (*W*) that are subject to upper-level decisions (*Y*).

Examples:

- ***Process design (upper-level) and process operation (lower-level)***
- ***Long-term planning (upper-level) and short-term scheduling (lower-level)***

NGDP Formulations

Traditional GDP does not consider nested disjunctions and requires transforming model into an Equivalent Single-level GDP (*Approach 1*):

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$

Single-Level Disjunctions

$$\bigvee_{j \in J_i} \begin{bmatrix} Y_{ij} \\ g_{ij}(x) \le 0 \end{bmatrix}$$

Extract nested disjunction (requires creating a new Boolean variable W_{ijk0})

$$\forall i \in I$$

$$\bigvee_{l \in L_{ijk}} \begin{bmatrix} W_{ijkl} \\ h_{ijkl}(x) \le 0 \end{bmatrix} \bigvee \begin{bmatrix} W_{ijk0} \\ x^{lo} \le x \le x^{up} \end{bmatrix}$$

$$\forall i \in I, j \in J_i, k \in K_{ij}$$

Logical Propositions

$$\bigvee_{j \in J_i} Y_{ij} \qquad \forall i \in I$$

New XOR proposition

$$\bigvee_{l \in L_{ijk}} W_{ijkl} \bigvee W_{ijk0} \qquad \forall i \in I, j \in J_i, k \in K_{ij}$$

$$\bigvee_{l \in L_{ijk}} W_{ijkl} \Leftrightarrow Y_{ij} \qquad \forall i \in I, j \in J_i$$

$$\Omega(Y, W) = True$$

Variable Domains

$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y_{ij} \in \{True, False\} \qquad \forall i \in I, j \in J_i$$

$$W_{ijkl} \in \{True, False\} \qquad \forall i \in I, j \in J_i, k \in K_{ij}, l \in L_{ijk} \cup \{0\}$$

The proposed approach is to explicitly model GDPs with the nested disjunctions (*Approach 2*):

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$

Nested Disjunctions

$$\bigvee_{j \in J_i} \left[\begin{array}{c} Y_{ij} \\ g_{ij}(x) \le 0 \\ \bigvee_{l \in L_{ijk}} \begin{bmatrix} W_{ijkl} \\ h_{ijkl}(x) \le 0 \end{bmatrix} \end{array} \right] \quad \forall k \in K_{ij} \qquad \forall i \in I$$

Logical Propositions

$$\bigvee_{j \in J_i} Y_{ij} \qquad \forall i \in I$$

$$\bigvee_{l \in L_{ijk}} W_{ijkl} \Leftrightarrow Y_{ij} \qquad \forall i \in I, j \in J_i$$

$$\Omega(Y, W) = True$$

Variable Domains

$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y_{ij} \in \{True, False\} \qquad \forall i \in I, j \in J_i$$

$$W_{ijkl} \in \{True, False\} \qquad \forall i \in I, j \in J_i, k \in K_{ij}, l \in L_{ijk}$$

NGDP Reformulation Approach 1 (Equivalent Single-level GDP Reformulation)

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0 \\ \bigvee_{k\in K_j}\begin{bmatrix} W_{jk} \\ h_{jk}(x)\le 0\end{bmatrix}\end{bmatrix}$$

Extract Inner Disjunction

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0\end{bmatrix}$$

$$\bigvee_{k\in K_j}\begin{bmatrix} W_{jk} \\ h_{jk}(x)\le 0\end{bmatrix}\bigvee\boxed{\begin{bmatrix} W_{j0} \\ x^{lo}\le x\le x^{up}\end{bmatrix}}\quad\forall j\in J$$

additional disjunct

The additional disjunct will incur the introduction of a <u>new binary variable</u> (ω_{j0}). A <u>new disaggregated variable</u> (μ_{j0}) is also required in the hull reformulation

Big-M Reformulation (BM1)

$$g_j(x)\le M_j(1-\lambda_j)\quad\forall j\in J$$

$$h_{jk}(x)\le m_{jk}(1-\omega_{jk})\quad\forall j\in J, k\in K_j$$

$$x^{lo}-\mathcal{M}_{jk}(1-\omega_{j0})\le x\le x^{up}+\mathcal{M}_{jk}(1-\omega_{j0})$$

Hull Reformulation (HR1)

$$x=\sum_{j\in J}\nu_j$$

$$\lambda_j g_j\left(\frac{\nu_j}{\lambda_j}\right)\le 0\quad\forall j\in J$$

$$x=\sum_{k\in K_j}\mu_{jk}+\boxed{\mu_{j0}}$$

additional slack

$$\omega_{jk}h_{jk}\left(\frac{\mu_{jk}}{\omega_{jk}}\right)\le 0\quad\forall j\in J, k\in K_j$$

NGDP Reformulation Approach 2 (Direct Nested GDP Reformulation)

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0 \\ \bigvee_{k\in K_j}\begin{bmatrix} W_{jk} \\ h_{jk}(x)\le 0 \end{bmatrix} \end{bmatrix}$$

Hull of Inner Disjunction

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0 \\ x=\sum_{k\in K_j}\mu_{jk} \\ \omega_{jk}h_{jk}\left(\frac{\mu_{jk}}{\omega_{jk}}\right)\le 0 \quad \forall j\in J, k\in K_j \end{bmatrix}$$

Big-M of Inner Disjunction

$$\bigvee_{j\in J}\begin{bmatrix} Y_j \\ g_j(x)\le 0 \\ h_{jk}(x)\le m_{jk}\cdot(1-\omega_{jk}) \quad \forall k\in K_j \end{bmatrix}$$

Hull Reformulation (HR2)

$$x=\sum_{j\in J}\nu_j$$
$$\lambda_j g_j\left(\frac{\nu_j}{\lambda_j}\right)\le 0 \quad \forall j\in J$$
$$\nu_j=\sum_{k\in K_j}\mu_{jk}$$
$$\omega_{jk}h_{jk}\left(\frac{\mu_{jk}}{\omega_{jk}}\right)\le 0 \quad \forall j\in J, k\in K_j$$

Inner disjunction variables are disaggregated only once

Big-M Reformulation (BM2)

$$g_j(x)\le M_j(1-\lambda_j) \quad \forall j\in J$$
$$h_{jk}(x)\le m_{jk}\cdot(1-\omega_{jk}) + M_j(1-\lambda_j) \quad \forall j\in J, k\in K_j$$

2 big-M parameters:
- m: inner disjunction
- M:outer disjunction

Big-M Reformulation Tightness

Theorem 1. The continuous relaxation of the **Direct NGDP Big-M Reformulation** (Approach 2 Big-M Reformulation; *r-BM2*) is as tight as the continuous relaxation of the **Equivalent Single-level GDP Reformulation** (Approach 1 Big-M Reformulation; *r-BM1*) when the tightest big-M values are used:

$$\textit{r-BM2} \subseteq \textit{r-BM1}$$

Proof: The two approaches differ in the Big-M reformulation of the constraints in the inner disjunct:

Approach 1: $h_{jk}(x) \leq m_{jk}\left(1-\omega_{jk}\right) \quad \forall j \in J, k \in K_j$

Approach 2: $h_{jk}(x) \leq m'_{jk} \cdot \left(1-\omega_{jk}\right) + M'_j\left(1-\lambda_j\right) \quad \forall j \in J, k \in K_j$

Let $h_{jk}^{max} = \max\left\{h_{jk}(x) | x \in R^n, x^{lo} \leq x \leq x^{up}\right\}$ be the maximum value of the constraint $h_{jk}(x)$ in the feasible space of x. The tightest Big-M values satisfy:

Approach 1: $m_{jk} = h_{jk}^{max}$

Approach 2: $m'_{jk} + M'_j = h_{jk}^{max}$

Since $\lambda_j \geq \omega_{jk}$, the right-hand side of the Big-M constraint in Approach 2 can be shown to be as tight as the right-hand side of the Big-M constraint in Approach 1:

$$m'_{jk} \cdot \left(1-\omega_{jk}\right) + M'_j\left(1-\lambda_j\right) \leq m_{jk}\left(1-\omega_{jk}\right)$$

Hull Reformulation Tightness

Theorem 2. The continuous relaxation of the **Direct NGDP Hull Reformulation** (Approach 2 Hull Reformulation; *r-HR2*) is as tight as the continuous relaxation of the **Equivalent Single-level GDP Reformulation** (Approach 1 Hull Reformulation; *r-HR1*):

r-HR2 ⊆ r-HR1

Proof: Use Fourier-Motzkin to eliminate the additional disaggregated variable (μ_{j0}) and its associated binary (ω_{j0}) resulting from the additional disjunct (W_{j0}) created when extracting the inner disjunct in Approach 1. Note the following,

- W_{j0} is selected iff the main disjunct is not (Y_j): $\omega_{j0} = 1 - y_j$.
- μ_{j0} is bounded by $0 \leq \mu_{j0} \leq x^{up}\omega_{j0}$.

The elimination yields the constraint,

$$\nu_j + \sum_{j' \in J : j' \neq j} \nu_{j'} - x^{up}(1 - y_j) \leq \sum_{k \in K_j} \mu_{jk} \leq \nu_j + \sum_{j' \in J : j' \neq j} \nu_{j'}$$

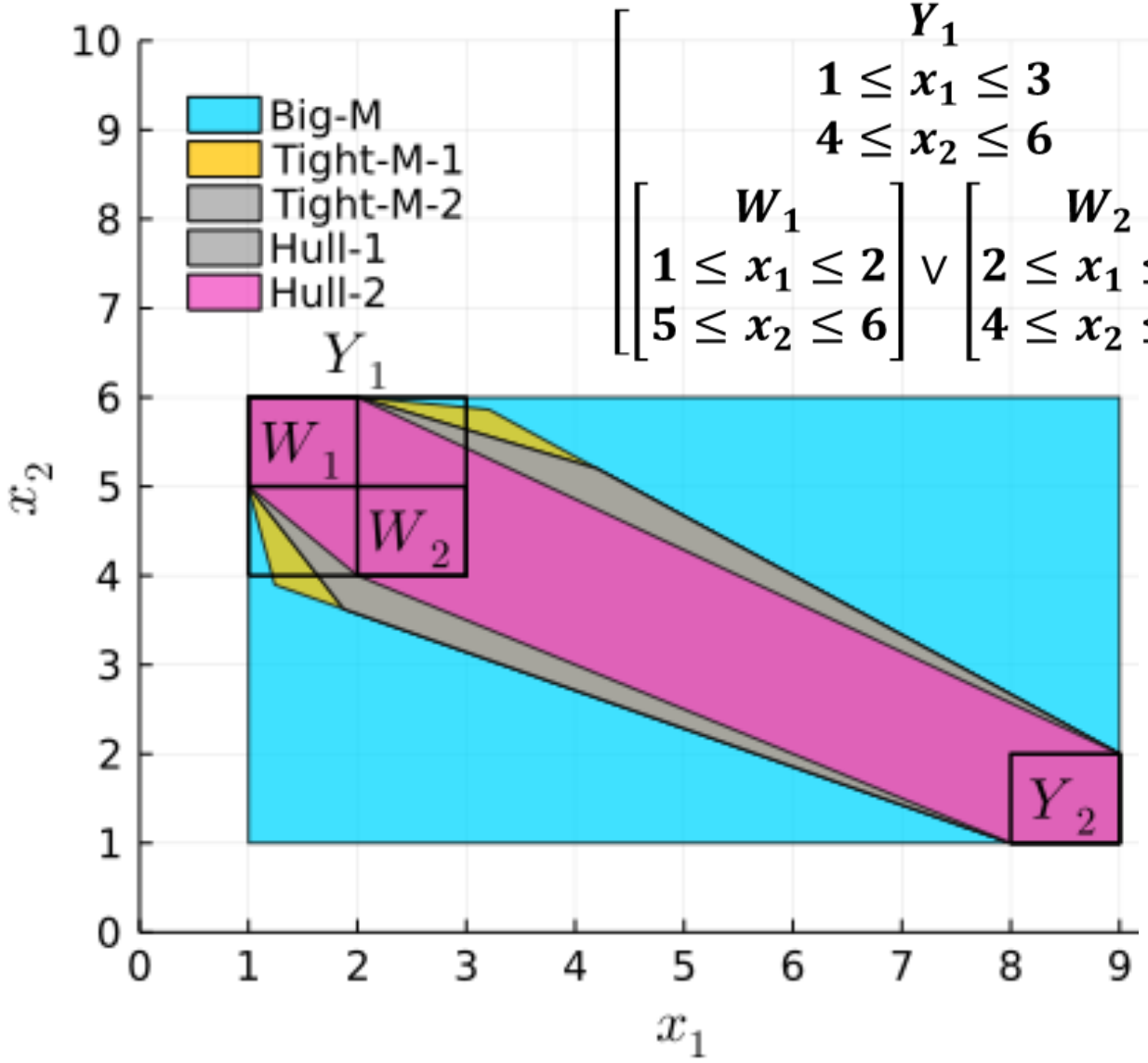
which can be shown to be a relaxation of the analogous constraint in Approach 2,

$$\nu_j \leq \sum_{k \in K_j} \mu_{jk} \leq \nu_j$$

Note: $0 \leq \sum_{j' \in J : j' \neq j} \nu_{j'} \leq x^{up}(1 - y_j)$

Reformulation Tightness Example

$$\begin{bmatrix} Y_1 \\ 1 \le x_1 \le 3 \\ 4 \le x_2 \le 6 \\ \begin{bmatrix} W_1 \\ 1 \le x_1 \le 2 \\ 5 \le x_2 \le 6 \end{bmatrix} \vee \begin{bmatrix} W_2 \\ 2 \le x_1 \le 3 \\ 4 \le x_2 \le 5 \end{bmatrix} \end{bmatrix} \vee \begin{bmatrix} Y_2 \\ 8 \le x_1 \le 9 \\ 1 \le x_2 \le 2 \end{bmatrix}$$



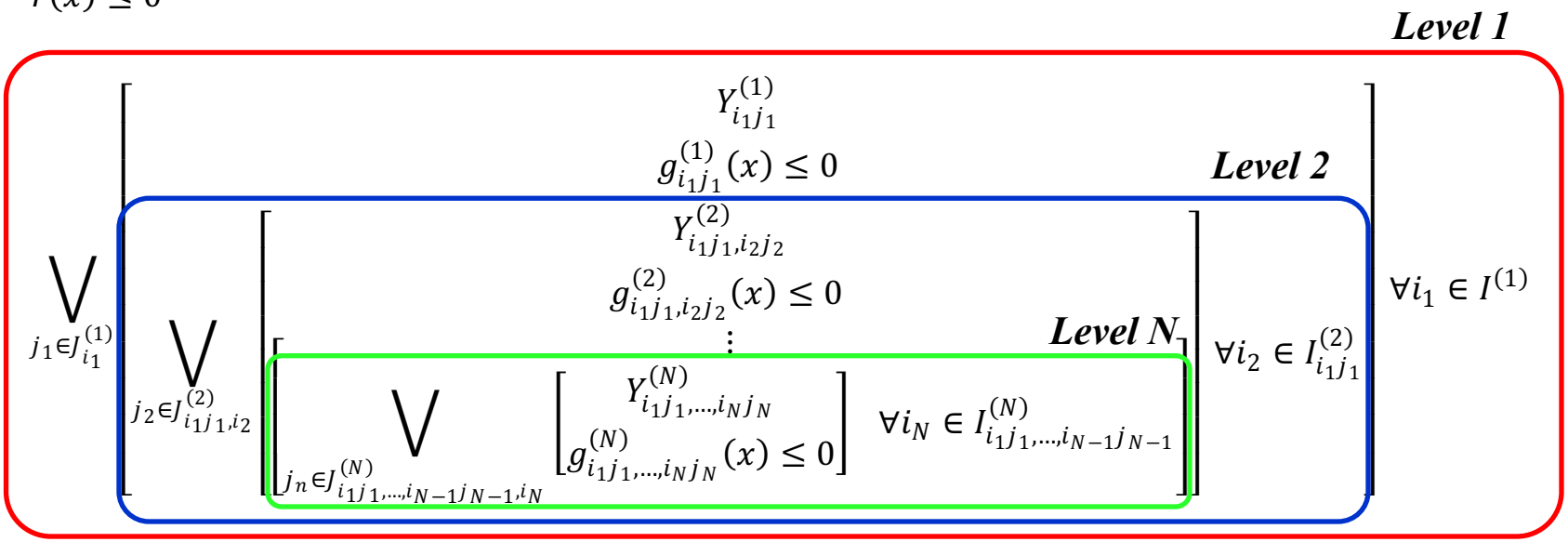
- **Big-M** = Big-M reformulation for Approach 1 or Approach 2
- **Tight-M-1** = Big-M reformulation for Approach 1 with tightest M values
- **Tight-M-2** = Big-M reformulation for Approach 2 with tightest M values
- **Hull-1** = Hull reformulation for Approach 1
- **Hull-2** = Hull reformulation for Approach 2

Remark

NGDPs and their reformulations to MI(N)LPs can be extended to **multi-level hierarchies**:

$$\min \quad Z = f(x)$$

$$s.t. \quad r(x) \le 0$$



$$\Omega(Y) = True$$

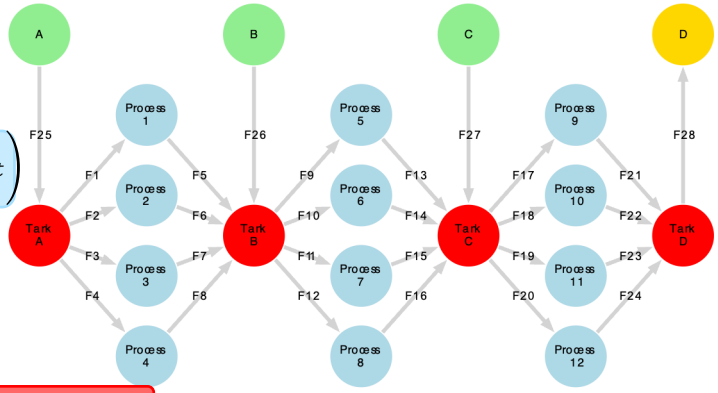
$$x^{lo} \le x \le x^{up}$$

$$x \in \mathbb{R}^n$$

$$Y^{(n)}_{i_1 j_j, \dots, i_k j_k} \in \{True, False\} \quad \forall n \in \{1, \dots, N\}$$

Illustrative Example

$$\max \sum_{t \in Time} \left(Revenue_{D,t} - \sum_{m \in \{A,B,C\}} ProcureCost_{m,t} - \sum_{i \in Processes} OperateCost_{i,t} \right) - \sum_{k \in Equipment} InstallCost_k$$



Superstructure optimization + production scheduling with multiple technology selection

***1** unit for each equipment type | **30** day horizon*

$$Level_{k,t} = Level_{k,t-1} + FlowIn_{k,t} - FlowOut_{k,t} \quad \forall k \in Tanks, t \in Time$$

$$ReactAvail_t = ReactAvail_{t-1} + \sum_{i \in Processes} \left(ReactConsumed_{i,t} - ReactConsumed_{i,t-\tau_i} \right) \quad \forall t \in Time$$

$$\bigvee_{u \in NumEquip_k} \begin{bmatrix} Y_{k,u}^{install} \\ InstallCost_k \end{bmatrix} \quad \forall k \in Equipment$$

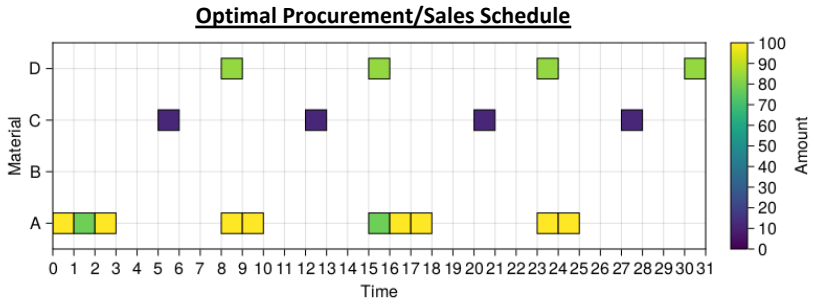
$$\begin{bmatrix} Y_i^{install} \\ ProductionCapacity_{i,t} \quad \forall t \\ \bigvee_{j \in Tech} \begin{bmatrix} W_{ij}^{technology} \\ TechCost_{i,t} \quad \forall t \\ ProductYield_{i,t} \quad \forall t \end{bmatrix} \\ \bigvee_{u \in NumReact} \begin{bmatrix} N_{i,u,t}^{trigger} \\ ReactConsumed_{i,t} \\ BatchSize_{i,t} \end{bmatrix} \quad \forall t \end{bmatrix} \veebar \left[\neg Y_i^{install} \right] \quad \forall i \in Processes$$

$$Y_i^{install} \Leftrightarrow \bigvee_{j \in Tech} W_{i,j}^{technology} \quad \forall i \in Processes$$

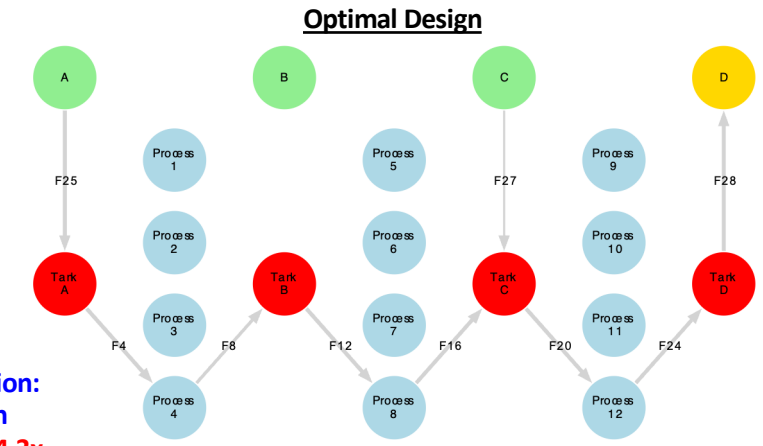
$$Y_i^{install} \Leftrightarrow \bigvee_{u \in NumEquip} N_{i,u,t}^{trigger} \quad \forall i \in Processes, t \in Time$$

$$N_{i,u,t}^{trigger} \Rightarrow \bigvee_{u' \geq u} Y_{React,u'}^{install} \quad \forall i \in Process, u \in NumReact, t \in Time$$

Illustrative Example Results



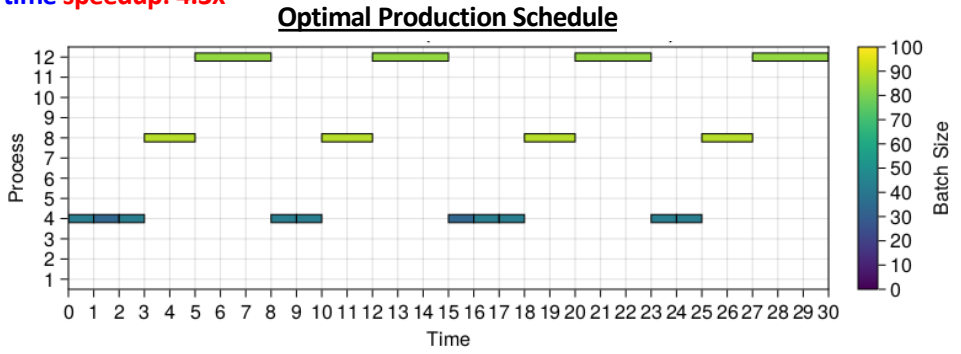
JuMP 1.3 (Julia 1.8)
DisjunctiveProgramming 0.3
CPLEX 20.1 (no presolve; no heuristics)
Optimal Profit: $20,434



	Hull Reformulation	
	Approach 1	*Approach 2*
Objective Function		
Relaxed MIP Solution	$701,518	$21,521
MIP Solution	$20,434	$20,434
Relaxation Gap	3333%	5%
MIP Gap	0%	0%
Nodes Explored	551,927	108,165
Cuts	39	45
CPU Time (s)	2,559	592
Model Size		
Binary Variables	1,198	814
Continuous Variables	14,742	12,536
Constraints	71,321	63,935

Nested GDP *(Approach 2)*:

- **Significantly tighter relaxation**
- **More compact formulation: 32%/15%/10% reduction**
- **Solution time speedup: 4.3x**



Optimization model for power generation and transmission expansion planning

Given: Generation sources, load demand for each region, CO_2 emission limits and economic data


Coal power plants


Natural gas power plants


Nuclear power plants


Wind turbines


Solar panels

***Determine*:** Number and type of generators / transmission lines, Unit commitments, Total cost

***Goal*: Long term Planning to Minimize Total Cost**

Method: Large-scale multiperiod MILP model (Benders decomposition)

Goal:

Account for Reliability (i.e., generators or transmission lines failures)

Generation: Lara, C. L., Mallapragada, D. S., Papageorgiou, D. J., Venkatesh, A., & Grossmann, I. E. (2018). Deterministic electric power infrastructure planning: Mixed-integer programming model and nested decomposition algorithm. *European Journal of Operational Research*, 271, 1037-1054.

Generation and Transmission: Li, C., A.J. Conejo, P. Liu, B.P. Omell, J.D. Siirola, I.E. Grossmann. (2021) Mixed-integer Linear Programming Models and Algorithms for Generation and Transmission Expansion Planning of Power Systems. *European Journal of Operational Research,* 297, 1071-1082.

Problem statement

Seolhee Cho

Goal

Develop a Generalized Disjunctive Programming model for multi-period and multi-site capacity planning of reliable power generation systems

Characteristics of the model

- Dual role of backup generators depending on power demand
- Advanced probabilistic model for reliability evaluation considering redundancy

Given

- Regions, power stations, backup generators
- Economic data (investment & operating cost)
- Unit reliability and electricity demand
- CO_2 emission and carbon tax

Determine

- When and where to install and retire the generators?
- Number and size of backup generators
- System reliability & expected power production

Overview of GDP optimization model

Summary of constraints

- **Balance of power plant *k*** that are available, decommissioned, and extend lifetime
- Balance and capacity of newly installed and available backup generator *j* in time *t*
- Available backup generators *j* in year *t* can either operate or remain as backup during sub-period *n* of year *t*
- Total fuel, total expected power output, CO_2 emission, and symmetry breaking constraints
- LOLE and LOEE penalties calculation
- **Failure state probability, and corresponding expected power output** under specific design *h* and operation mode *m*
  - → These are calculated based on **embedded disjunctions**
  - → Two Boolean variables: $Z_{k,r,h,t}$ – Planning decision, $W_{k,r,m,h,t,n}$ – Operation decision

Nested Disjunction in GDP formulation

$$\left[\bigvee_{c\in C_k}\begin{bmatrix} X^{OP}_{j,k,r,t,n} \\ X^{BOP}_{j,c,k,r,t,n} \\ OC^{BAK}_{j,c,k,r,t,n} = \psi_{j,c,k} C^{BOP}_{j,c,k,r,t,n} \\ \gamma^{min}_{j,k}\varphi_{j,c,k} \le C^{BOP}_{j,c,k,r,t,n} \le \gamma^{max}_{j,k}\varphi_{j,c,k}\end{bmatrix}\right] \vee \begin{bmatrix} \neg X^{OP}_{j,k,r,t,n} \\ OC^{BAK}_{j,c,k,r,t,n} = 0 \\ C^{BOP}_{j,c,k,r,t,n} = 0\end{bmatrix}$$

$$\forall j \in J_k, k \in K_r, r \in R, t \in T, n \in N$$

$$\begin{bmatrix} X^{PO}_{k,r,t,n} \\ OC^{P}_{k,r,t,n} = \delta_k C^{PO}_{k,r,t,n} \\ \gamma^{min}_{k}\omega_k \le C^{PO}_{k,r,t,n} \le \gamma^{max}_{k}\omega_k \\ 0 \le P^{F}_{k,r,t,n} \le 1 \\ 0 \le EP_{s,k,r,t,n} \le \overline{M_k} \quad \forall s \in S\end{bmatrix} \vee \begin{bmatrix} \neg X^{PO}_{k,r,t,n} \\ OC^{P}_{k,r,t,n} = 0 \\ C^{PO}_{k,r,t,n} = 0 \\ P^{F}_{k,r,t,n} = 1 \\ EP_{s,k,r,t,n} = 0\end{bmatrix} \quad \forall k \in K_r, r \in R, t \in T, n \in N$$

$$X^{OP}_{j,k,r,t,n} \Leftrightarrow \bigvee_{c\in C_k} X^{BOP}_{j,c,k,r,t,n} \quad \forall j \in J_k, k \in K_r, r \in R, t \in T, n \in N$$

$$X^{BOP}_{j,c,k,r,t,n} \Rightarrow Y^{BAV}_{j,c,k,r,t} \quad \forall j \in J_k, c \in C_k, k \in K_r, r \in R, t \in T, n \in N$$

$$X^{BOP}_{j,c,k,r,t,n} \Rightarrow X^{PO}_{k,r,t,n} \quad \forall j \in J_k, c \in C_k, k \in K_r, r \in R, t \in T, n \in N$$

$$X^{PO}_{k,r,t,n} \Rightarrow Y^{PA}_{k,r,t} \quad \forall k \in K_r, r \in R, t \in T, n \in N$$

$$\bigvee_{h\in H_k}\left[\bigvee_{m\in M_h}\begin{bmatrix} EP_{s,k,r,t,n} = P_{s,k,r}\left(C^{PO}_{k,r,n,t} + \sum_{j\in J^O_{s,m,h}} C^{OP}_{j,k,r,t,n}\right) & \forall s \in S^F_{m,h} \\ EP_{s,k,r,t,n} = P_{s,k,r}\left(C^{PA}_{k,r,t} + \sum_{j\in J^O_{s,m,h}} C^{AV}_{j,k,r,t}\right) & \forall s \in S^P_{m,h}\end{bmatrix} \forall n \in N\right] \forall k \in K_r, r \in R, t \in T$$

$$P^F_{k,r,t,n} = \sum_{s\in S^F_{m,h}} P_{s,k,r}$$

$$P_{s,k,r} = \prod_{j\in J^A_{s,h}} \lambda_{j,k} \prod_{j\in J^F_{s,h}} (1-\lambda_{j,k}) \quad \forall s \in S_h, |S| = 2^{|h|}, k \in K_r, r \in R$$

$Z_{k,r,h,t}$ → Design decisions in time *t*

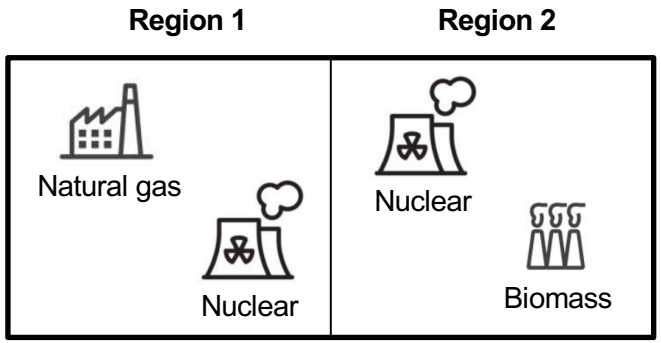
$W_{k,r,m,h,t,n}$ → Operation decisions during subperiod *n* in time *t*

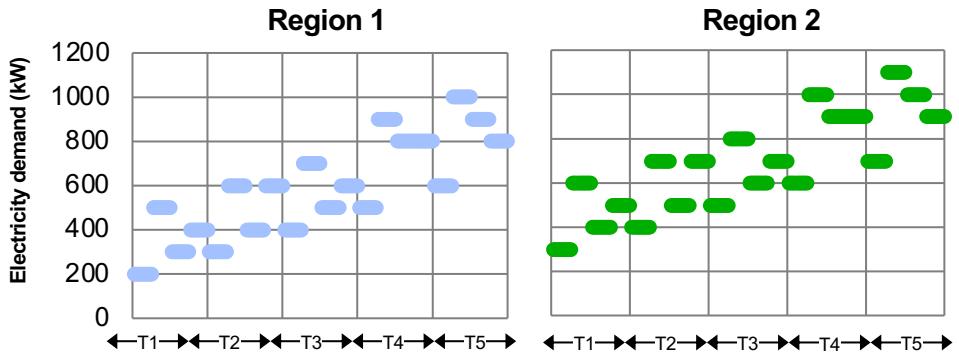
Successful reliability

Expected production level depending on design and operation

Illustrative example

- Two regions (R1, R2) and three different power technologies are installed **(natural gas, nuclear, and biomass gasification)**
- By adding backup generators, each power plant can both expand its capacity and improve reliability





	NG/Nuclear/Biomass	Three capacities for backup generators	small/medium/large
Unit reliability (%)	90 / 95 / 95	Capacity size of backup generator (kW)	100 / 200 / 300
Installed capacity of power plant (kW)	200 / 300 / 200	Installation cost of NG by size (k$/kW)	5 / 8 / 10
Operating cost ($/kW)	20 / 40 / 15	Installation cost of nuclear by size (k$)	15 / 20 / 25
Conversion efficiency (%)	40 / 45 / 40	Installation cost of biomass by size (k$)	2.5 / 4.5 / 6
Feedstock cost ($/MMBtu)	0.3 / 0.01 / 0.2	Average CO_2 emission cost ($/kg)	3
Operating cost of power plant ($/kW)	20 / 40 / 15		

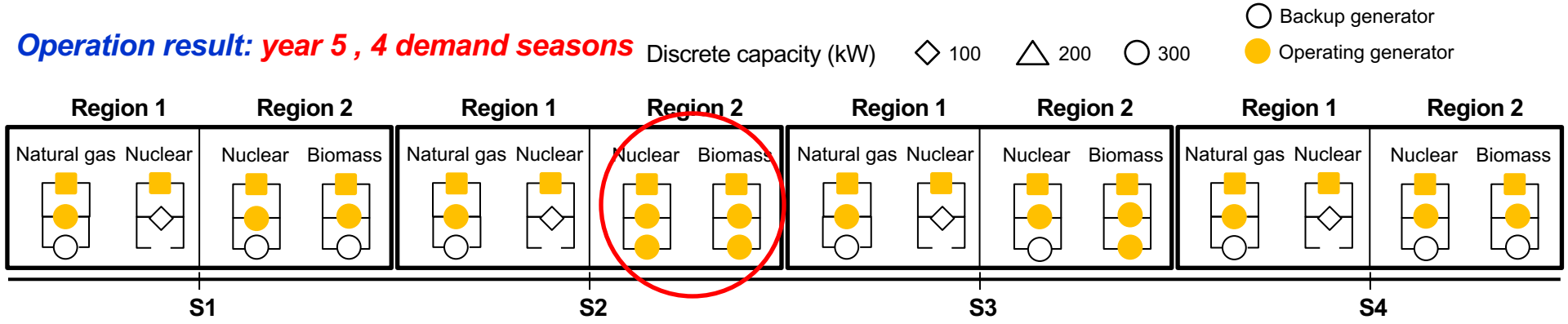
LOLE (unmet demand) penalty rate ($/kWh)	10	LOEE (downtime) penalty rate ($/hour)	5,000

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Illustrative example:

2 regions: Natural gas/Nuclear, Nuclear/Biomass over 5 year time horizon

Operation result: year 5 , 4 demand seasons



- **Backup generator are used to produce more power (Region 2/2nd demand period)**
- Reliable design and operation strategies are highly dependent by LOLE and LOEE penalties

Computational results

Approach 2	Constraints	Cont. Variables	Binary variables	CPU (sec)	Rel gap%	LP relaxation (k$)	Cost (k$)
Big-M	12,419	4,631	1,644	1,668.31	84.0	115.7	722.21
Hull reformulation	23,503	12,435	1,644	10.84	11.8	636.9	722.21

- Solver: Gurobi 32.1.0
- Tight bounds of convex hull reformation → Short computational time

Conclusions

- Extending GDP to allow ***nested disjunctions*** allows explicitly modeling problems with multiple hierarchies
- Two approaches to reformulate NGDPs into MI(N)LPs have been formalized:
  - Approach 1: transforming into an equivalent single-level GDP and then reformulating
  - Approach 2: reformulating NGDP from the inside-out
- Relaxations of reformulated models via Approach 2 are shown to be as tight or tighter than their counterparts obtained via Approach 1
- Examples show superiority Approach 2 *(Nested)*