Strong formulations for (mixed) integer quadratic modeling (M)IQM

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Outline

- (M)IQCQP, (mixed) integer quadratic constrained quadratic programming
- (M)IQM types
- Aim
- Alternative modeling objects:
  - Impulse-variables based
  - Step-variables based
- Realistic application case studies under uncertainty: EGTCEP, VTAC, RTND, ENDO, SNDP
- References and bibliography

$$min\, a^t z + b^t x + y^t Cx$$

$$\text{s.t. } Az + Bx + y^t Dx = c$$
$$z \in \mathbb{R}^{n1},\ x \in \mathbb{Z}_+^{n2},\ 0 \leq y \leq \overline{y}^{n3}.$$

Note: For our purposes, variables vector y can have integer elements, wlog.

Type 1 of (M)IQM $y_t x_{t'}$, for $t', t \in \mathcal{T} : t' \leq t$

Dependency of each other vars (x,y):
$y_t > 0 \rightarrow x_{t'} > 0$ and $x_{t'} = 0 \rightarrow y_t = 0$. Examples:

- Production (*tactical/operational* decisions) in a facility network w/
  location & capacity expansion (*strategic* decisions).
- Distribution hub location and expansion.
- Energy generation & transmission capacity expansion planning.
- Supply network management in automation and assembly.
- Rapid transit network design and other facilities.
- Distributionally robust optimization in multi-horizon stochastic environments DRO-MHSE (work-in-progress) .

Type 2 of (M)IQM $y\ x$

Independency of each other vars (x,y):
$y > 0 \nrightarrow x > 0$ and $x = 0 \nrightarrow y = 0$. Examples:

- Cross-dock door assignment problem CDAP,
  Guignard, Hahn, Pessoa, Cardoso da Silva Procs 2012;
  Guignard ANOR 2020;

  LFE, Garín, Unzueta IJPR 2023

  by using Adams, Sherali OR 1990 RLT1 scheme.
- Piecewise MIQM approx of nonlinear functions:
  - Aircraft conflict resolution via VTAC: horizontal (velocity and angle turn) and vertical (altitude level) maneuvers.
    Alonso-Ayuso, LFE, MartínCampo EJOR 2018.
  - Disaster relief mgmnt via ENDO.
    LFE, Garín, Monge, Unzueta COR 2018, EJOR 2020.

Aim

Seeking for a strong & computationally tractable modeling;
Mathematically equivalent linearizations of (M)IQM terms.

- Each other dependent vars: Alternative modeling objects
  - *Impulse variable*,
    taking benefit of the inherent S1 induced structure
    á là Kolodziej, Castro, Grossmann JOGO 2013.
  - *Step-variable*
    Bertsimas, StockPatterson OR 1998; Alonso-Ayuso, LFE TOP 1998;
    Alonso-Ayuso, LFE, Garín, Ortuño, Pérez JOGO 2003; and others.
- Independent vars modeling object
  - *Impulse variable*
    Alonso-Ayuso, LFE, MartínCampo EJOR 2018;
    LFE, Garín, Monge, Unzueta COR 2018, EJOR 2020.
- Balancing
  pros (strong modeling) and cons (dimensions increasing).

Form alternative #1: (M)IQM term $y_t x_{t'}$, for $t', t \in \mathcal{T} : t' \leq t$, can be replaced with var y_t alone plus an impulse-var based upper bound

$x_{t'}$, a **binary impulse**-var representing the occurrence of the event **at** stage t', such that it can only occur once, at most, along the caligraphically ordered horizon, for $t' \in \mathcal{T}$.

Application field: multistage time horizon

That is, $\{x_{t'} \, \forall t' \in \mathcal{T}\}$: special ordered set S1

Beale, book chap 1970

$$
\begin{aligned}
& x_t \in \{0,1\}, \sum_{t' \in \mathcal{T}} x_{t'} \leq 1 \\
& 0 \leq y_t \leq \overline{y}_t \sum_{t' \in \mathcal{T} : t' \leq t} x_{t'}
\end{aligned}
\tag{1}
$$

Form alternative #2: (M)IQM term $y_t x_{t'}$, for $t', t \in \mathcal{T} : t' \leq t$, can be replaced with new var $f_{t',t}$ plus a Fortet linear system

Fortet inequalities RFRO 1960;
Hammer, Rudeanu 1968 book

(also known as McCormick constraints MP 1976).

For $t \in \mathcal{T}$:

$$
\begin{aligned}
& x_t \in \{0,1\}, \sum_{t' \in \mathcal{T}} x_{t'} \leq 1 & & \\
& 0 \leq f_{t't} \leq \overline{y}_t x_{t'} & & \forall t' \in \mathcal{T} : t' \leq t \\
& f_{t',t} \leq y_t & & \forall t' \in \mathcal{T} : t' \leq t \\
& y_t - f_{t't} \leq \overline{y}_t (1 - x_{t'}) & & \forall t' \in \mathcal{T} : t' \leq t
\end{aligned}
\tag{2}
$$

Impulse-var based #2 **same strength** as impulse-var based #1.

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& y_t - f_{t't} \leq \overline{y}_t (1 - x_{t'}) & & \forall t' \in \mathcal{T} : t' \leq t
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Impulse-var based #2 **same strength** as impulse-var based #1.

Form alternative #3: (M)IQM term $y_t x_{t'}$, for $t', t \in \mathcal{T} : t' \leq t$, can be replaced with new var $f_{t',t}$ plus a Fortet linear system that takes benefit of the S1 induced constraint

For $t \in \mathcal{T}$:

$$
\begin{aligned}
& x_t \in \{0,1\}, \sum_{t' \in \mathcal{T}} x_{t'} \leq 1 \\
& 0 \leq f_{t',t} \leq \overline{y}_t x_{t'} \quad \forall t' \in \mathcal{T} : t' \leq t \\
& \sum_{t' \in \mathcal{T} : t' \leq t} f_{t',t} \leq y_t \quad \left(\text{instead of } f_{t',t} \leq y_t \, \forall t' \in \mathcal{T} : t' \leq t\right) \\
& y_t - f_{t',t} \leq \overline{y}_t (1 - x_{t'}) \quad \forall t' \in \mathcal{T} : t' \leq t
\end{aligned}
\tag{3}
$$

It is **stronger** than impulse-var based #1, #2.

Form alternative #3: (M)IQM term $y_t x_{t'}$, for $t', t \in \mathcal{T} : t' \leq t$, can be replaced with new var $f_{t',t}$ plus a Fortet linear system that takes benefit of the S1 induced constraint

For $t \in \mathcal{T}$:

$$
\begin{aligned}
& x_t \in \{0,1\}, \sum_{t' \in \mathcal{T}} x_{t'} \leq 1 \\
& 0 \leq f_{t',t} \leq \overline{y}_t x_{t'} \quad \forall t' \in \mathcal{T} : t' \leq t \\
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& y_t - f_{t',t} \leq \overline{y}_t (1 - x_{t'}) \quad \forall t' \in \mathcal{T} : t' \leq t
\end{aligned}
\tag{3}
$$

It is **stronger** than impulse-var based #1, #2.

á là Kolodziej, Castro, Grossmann JOGO 2013.

$$x_{t'} \in \{0,1\}\, \forall t \in \mathcal{T}, \sum_{t' \in \mathcal{T}} x_{t'} \leq 1$$

$$y_t \sum_{t' \in \mathcal{T}: t' \leq t} x_{t'} \leq y_t \Rightarrow \sum_{t' \in \mathcal{T}: t' \leq t} f_{t',t} \leq y_t$$

where $f_{t',t} \equiv y_t x_{t'}\, \forall t' \in \mathcal{T} : t' \leq t,\ t \in \mathcal{T}$

Form alternative #4: (M)IQM term $y_t x_{t'}$, for $t', t \in \mathcal{T} : t' \leq t$, can be replaced with var y_t alone plus a step-var based upper bounding

x_t, a **integer step**-var representing the occurrence of the event **by** stage t, for $t \in \mathcal{T}$,

Application field: multistage time horizon

For $t \in \mathcal{T}$:

$$\begin{aligned} & x_t \in \{0,1\},\ x_{t-1} \leq x_t \\ & 0 \leq y_t \leq \overline{y}_t x_t \end{aligned} \tag{4}$$

It is **stronger** than impulse-var based #1, #2.
E.g., cases for 0-1 values:

- $x_t - x_{t-1} = 1$: The event occurs *at* stage t.
- $x_t - x_{t-1} = 0$:
  Either it did occur *by* stage $t-1$ (i.e., $x_{t-1} = 1$)
  or it *did not* occur yet.

Form #5: (M)IQM term $y_t\, x_t$, for $t \in \mathcal{T}$, can be replaced with new var f_t plus a Fortet linear system that takes benefit of the S1 induced constraint

x_t, a **binary impulse**-var to allow only one var $y_t > 0$, at most, for $t \in \mathcal{T}$.

piecewise MIQM approx of nonlinear functions.

$$
\begin{aligned}
& x_t \in \{0,1\} \forall t \in \mathcal{T}, \sum_{t \in \mathcal{T}} x_t \leq 1 \\
& 0 \leq f_t \leq \overline{y}_t x_t \quad \forall t \in \mathcal{T} \\
& \sum_{t \in \mathcal{T}} f_t \leq y_t \\
& y_t - f_t \leq \overline{y}_t (1 - x_t) \quad \forall t \in \mathcal{T}
\end{aligned}
\tag{5}
$$

Impulse-var based #5:
a particular case of impulse-var based #3, where $t' = t$.

Summary and a sort of evaluation

- A (M)IQM term is replaced with the y-var or a new one plus a linear constraint system.
  Impulse-var form allows to use inherent S1 induced info.
- *Step*-var form: Advantage for multistage model decom: x-state vars only intersect two consecutive stages.
- Our favorite linearization modeling:

#4, x step-var based $y_t x_t$:

$$x_t \in \{0,1\},\ x_{t-1} \leq x_t,\ 0 \leq f_t \leq \overline{y}_t x_t$$

#3, x impulse-var based $y_t x_{t'}$:

$$x_{t'} \in \{0,1\},\ \sum_{t' \in \mathcal{T}} x_{t'} \leq 1,\ 0 \leq f_{t',t} \leq \overline{y}_t x_{t'},$$

$$\sum_{t' \in \mathcal{T}: t' \leq t} f_{t',t} \leq y_t,\ y_t - f_{t',t} \leq \overline{y}_t (1 - x_{t'})$$

#5, x impulse-var based $y_t\, x_t$:

$$x_t \in \{0,1\},\ \sum_{t \in \mathcal{T}} x_t \leq 1,\ 0 \leq f_t \leq \overline{y}_t x_t,$$

$$\sum_{t \in \mathcal{T}} f_t \leq y_t,\ y_t - f_t \leq \overline{y}_t (1 - x_t)$$

Realistic application case studies under uncertainty: EGTCEP

Energy Generation & Transmission Capacity Expansion Planning w/ **step-var** based form #4.

- Estimating the appropriate *feasible type and mix of power generation sources* (ranging from less coal, nuclear and combined cycle gas turbine to more renewable sources: hydroelectric, wind, solar, photovoltaic and biomass) w/ power generation plant / farm *location*, and capacity of new lines in the *transmission network*.
- A sequential MIQCQP approx of a large MINLP model:

LFE. *Workshop on Models and Methods for Energy Optimization*. Budapest, 2014.

Realistic application case studies under uncertainty: VTAC

Aircraft conflict resolution problem w/ **impulse-var** based form #5.

- Horizontal (velocity and angle turn changes) and vertical (altitude level changes) maneuvers.
- A multi-objective model based on a geometric construction which involves trigonometric functions.
- A sequential MIQCQP approx of a MINLP model: MINLP terms (say, velocity $\times$ nonlinear function of angle motion) are approximated based on angle rank dynamic discretization requiring only one angle of motion to be selected.

Alonso-Ayuso, LFE, MartínCampo EJOR 2018.

Realistic application case studies under uncertainty: RTNDP

Rapid Transit Network Design Planning w/ **step-var** based form #4:

- Strategic decisions about the timing to construct stations and edges, and
  operational decisions on the available network at the stages in the time horizon.
- A mixture of fix-and-relax and lazy matheuristic algos:
  A soln with a good optimality gap and
  an affordable elapsed time.
- A very large BQP model:
  2.6M constraints and 1.6M binary variables.

Cadarso, LFE, Marín EJOR 2018.

Realistic application case studies under uncertainty: ENDO

Endogenous uncertainty treatment for disaster relief mgmnt w/ **impulse-var** based form #5.

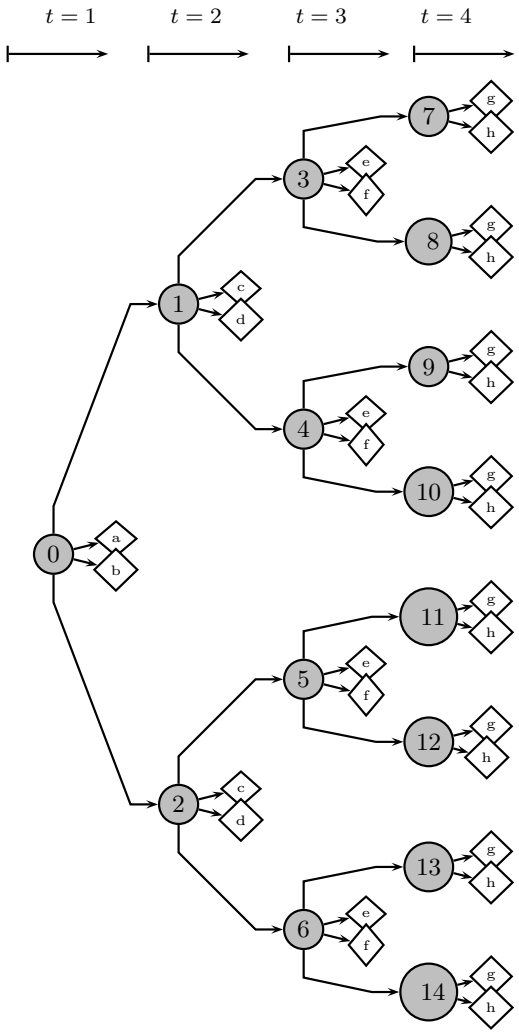
- Exogenous uncertainty, where the decision maker cannot influence it, but he must hedge his decisions against future values of the uncertain parameters.
- *Endogenous uncertainty* (ENDO), where the decision maker (usually, **investments**) can influence the weight of future values of some uncertain parameters (i.e., **scenarios**).
- A MIQCQP approx of a MINLP model: piecewise linear approx of nonlinear functions.

LFE, Garín, Monge, Unzueta COR 2018, EJOR 2020.

Realistic application case studies under uncertainty: SNDP

Supply Network Design Planning w/ **step-var** based form #4.

- Strategic decisions on the production topology: location of plants to open, plant sizing, product selection, product allocation among plants and vendor selection for key raw material.
- A very large MIQCQP model.
  Objective: max. of *expected net revenue plus risk mgmnt* along the time horizon.
  Alonso-Ayuso, LFE, Garín, Ortuño, Pérez JOGO 2003; Alonso-Ayuso, LFE, Ortuño book chap 2005; LFE, Monge, RomeroMorales (SDP-SD) COR15; LFE, Monge book chap. 2023 in press.
- Constructive algo SFR3: Scenario variables Fixing and iteratively Randomization of constraints and variables' integrality Relaxation Reduction, for distributionally robust MILP optimization with ECSD risk averse in multi horizon problems.
  Alonso-Ayuso, LFE, Monge, Morales, Pardo DRO-MHSE, work in progress.



$$\Pi_3 = \{e, f\}$$
$$\mathcal{N} = \{0, \cdots, 14\}$$
$$\mathcal{N}_2 = \{1, 2\}$$
$$\mathcal{A}^7 = \{0, 1, 3, 7\}$$
$$\mathcal{S}_1^1 = \{3, 4\}$$

Figura: Strategic multistage with embedded operational two-stage scenario trees

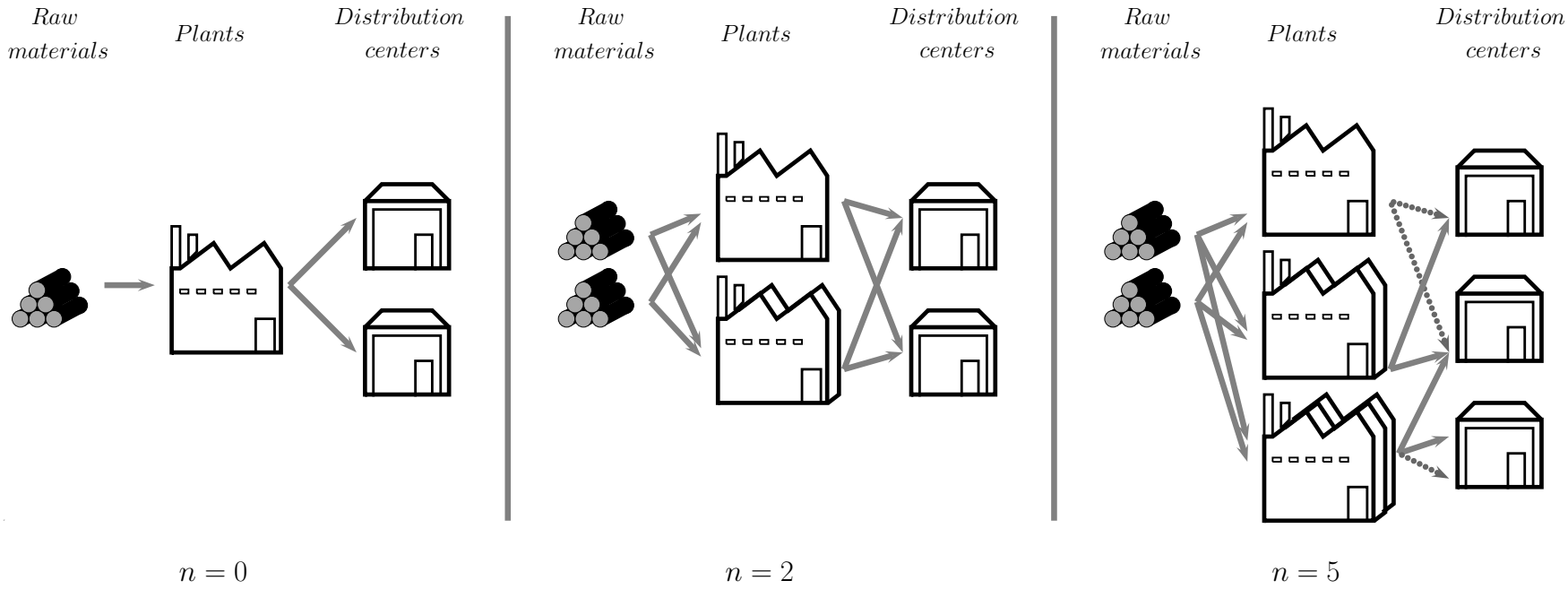


Figura: Supplying, manufacturing plants and distribution center network design and expansion

Realistic application case studies under uncertainty: Very large-scale SNDP

- BlockIP-MMSO: IPM for MSLP/MSQP
- Strategic multistage / operational two-stage scenario trees
- Structures: **step-var** based form #4 and block-diagonal matrix preconditioner
- Eqs solving: a combination of Cholesky factor and a PCG
- Broad computational experience:
  5 stages, 40 plants, 2000 raw material, 4000 DCs,
  3750 scenario tree nodes,
  839 million L vars, 13 million Q vars, 21 million cons.

Realistic application case studies under uncertainty: Very large-scale SNDP (c)

BlockIP-MMSO: IPM for MSLP/MSQP

- HW/SW: a DELL PowerEdge R7525, two 2.4 GHz AMD EPYC 7532 CPUs (128 total cores), RAM 768 GB, a GNU/Linux OS (openSuse 15.3).
- Computing time: #thread=1
  - BlockIP-MMSO: 1.1 days using RAM 174 GB.
  - CPLEX v20.1 (primal-dual barrier): 53 days and 531 GB.
- Computing time: #thread=100
  - BlockIP-MMSO: 4.18 hours

    (a non-sophisticated parallelization)
  - CPLEX v20.1 (primal-dual barrier): 35.6 hours.

Castro, LFE, Monge 2023 EJOR R2 (very minor changes).

References

- W.P. Adams, H-D. Sherali. Linearization strategies for a class of zero-one mixed integer programming problems. *Operations Research*, 38:217-226, 1990.
- A. Alonso-Ayuso, L.F. Escudero. 0-1 equivalent model representations for Air Traffic Management. *TOP*, 6:19-36, 1998.
- D. Bertsimas, S. Stock-Patterson, The air traffic flow management problem with enroute capacities. *Operations Research*, 46:406-422, 1998.
- R. Fortet. Application de l'álgebre de boole en recherche operationelle. *Revue Francaise de Recherche Operationelle*, 4:17-26, 1960.
- L.F. Escudero, M.A. Garín. A. Unzueta. On solving the Cross-dock Door Assignment Problem. *International Journal of Production Research*, doi.org/10.1080/00207543.2023.2180307, 2023.
- M. Guignard, P.M. Hahn, A.A. Pessoa, D. Cardoso da Silva. Algorithms for the Cross-dock Door Assignment Problem. *In Proceedings of the Fourth International Workshop on Model-Based Metaheuristics*, 2012.
- M. Guignard. Strong RLT1 bounds from decomposable Lagrangean Relaxation for some quadratic 0-1 optimization problems with linear constraints. *Annals of Operations Research*, 286:173-200, 2020.
- P.L. Hammer, S. Rudeanu. *Boolean methods in Operations Research and related areas*. Springer, 1968.
- S. Kolodziej, P.M. Castro, I.E. Grossmann. Global optimization of bilinear programs with a multiparametric disaggregation technique. *Journal of Global Optimization*, 57:1039-1063, 2013.
- G.P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I - Convex underestimating problems. *Mathematical Programming*, 10:147-175, 1976.

OTHER BIBLIOGRAPHY ON APPLICATIONS

- M. Albareda-Sambola, A. Alonso-Ayuso, L.F. Escudero, E. Fernández, Pizarro. On solving the multi-period location-assignment problem under uncertainty. *Computers & Operations Research*, 40:2878-2892, 2013.
- A. Alonso-Ayuso, L.F. Escudero, A. Garín, M.T. Ortuño, G. Pérez. A Stochastic 0-1 Program based approach for Strategic Supply Chain Planning under Uncertainty. *Journal of Global Optimization*, 26:97-124, 2003.
- A. Alonso-Ayuso, L.F. Escudero, M. Guignard, A. Weintraub. Risk management for forestry planning under uncertainty in demands and prices. *European Journal of Operational Research*, 267:1051-1074, 2018.
- A. Alonso-Ayuso, L.F. Escudero, M. Guignard, A. Weintraub. On dealing with strategic and tactical decision levels in forestry planning under uncertainty. *Computers & Operations Research*, 115:104836, 2020.
- A. Alonso-Ayuso, L.F. Escudero, F.J. Martin-Campo. Multiobjective optimization for aircraft conflict resolution. A metaheuristic approach. *European Journal of Operational Research*, 248:691-702, 2016.
- A. Alonso-Ayuso, L.F. Escudero, F.J. Martín-Campo. An exact multi-objective mixed integer nonlinear optimization approach for aircraft conflict resolution. TOP, 24:381-408, 2016.
- A. Alonso-Ayuso, L.F. Escudero, J.F. Monge, D. Morales, L. Pardo. On distributionally robust MILP optimization with risk averse in multi horizon problems. Work in progress.
- A. Alonso-Ayuso, L.F. Escudero, M.T. Ortuño. A stochastic 0-1 program based approach for Air Traffic Management. *European Journal of Operational Research*, 120:47-62, 2000.

- A. Alonso-Ayuso, L.F. Escudero, M.T. Ortuño. BFC, On modeling production planning and scheduling under uncertainty, in: S. Wallace, W.T. Ziemba (eds.). *Applications of Stochastic Programming. MPS-SIAM Series in Optimization*, pp. 217-252, 2005.
- A. Alonso-Ayuso, L.F. Escudero, M.T. Ortuño, On Modeling Planning under Uncertainty in Manufacturing. *SORT, Statistics and Operations Research Transactions*, 31:109-150, 2007.
- S. Baptista, A.P. Barbosa-Povoa, L.F. Escudero, M.I. Gomes, C. Pizarro. On risk management for a two-stage stochastic mixed 0-1 model for designing and operation planning of a closed-loop supply chain. *European Journal of Operational Research*, 274:91-107, 2019.
- L. Cadarso, L.F. Escudero, A. Marín. On strategic multistage operational two-stage stochastic 0-1 optimization for the Rapid Transit Network Design problem. *European Journal of Operational Research*, 271:577-593, 2018.
- J. Castro, L.F. Escudero, J.F. Monge. On solving multistage stochastic optimization problems with the Interior Point Method MSO-BlockIP. *European Journal of Operational Research R2*, 2023.
- L.F. Escudero. On modeling Energy Generation Capacity and Transmission Expansion Planning problem. An approximation to Multicrieria Risk Averse Multistage Stochastic MINLP. *Workshop on Models and Methods for Energy Optimization*. Budapest (Hungary), http://www.math.bme.hu/CostWG2014.html, 2014.
- L.F. Escudero, M.A. Garín, J.F. Monge, A. Unzueta. On preparedness resource allocation planning for natural disaster relief under endogenous uncertainty with time-consistent risk-averse management. *Computers & Operations Research*, 88:84-102, 2018.

- L.F. Escudero, M.A. Garín, J.F. Monge, A. Unzueta. Some matheuristic algorithms for multistage stochastic optimization models with endogenous uncertainty and risk management. *European Journal of Operational Research*, 285:988-1001, 2020.
- L.F. Escudero, M.A. Garín, J.F. Monge, A. Unzueta. On a two-stage stochastic approach for Cross-dock Door Design Problem. Work-in-progress.
- L.F. Escudero, M.A. Garín. A. Unzueta. On solving the Cross-dock Door Assignment Problem. *International Journal of Production Research*, doi.org/10.1080/00207543.2023.2180307, 2023.
- L.F. Escudero, J.F. Monge. On capacity expansion planning under strategic and operational uncertainties based on stochastic dominance risk averse management. *Computational Management Science*, 15:479-500, 2018.
- L.F. Escudero, J.F. Monge. On multistage multiscale stochastic capacitated multiple allocation hub network expansion planning. *Mathematics*, 9:3177, 2021.
- L.F. Escudero, J.F. Monge. On risk management of multi horizon FLP under uncertainty. Chapter in H.A. Eiselt, V. Marianov (eds.),Uncertainty in facility location models: Incorporating location science and randomnes. Springer-Verlag. Accepted for publication. Springer Nature, 2023.
- L.F. Escudero, J.F. Monge, A. Rodríguez-Chía. On pricing-based equilibrium for network expansion planning. A multi-period bilevel approach under uncertainty. *European Journal of Operational Research*, 287:262-279, 2020.
- L.F. Escudero, C. Pizarro. On solving a large-scale problem on facility location and customer assignment with interaction costs along a time horizon. *TOP*, 25:601-622, 2017.