Huff location models on networks

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Outline

1. Introduction
   - Definition of a location problem
   - A couple of famous examples of location problems on networks
   - Location problems in a competitive environment

2. Problem definition
   - Huff location model with continuous demand
   - Huff location model with discrete demand

3. Algorithms and results
   - VNS metaheuristic
   - Computational results

4. Conclusions and future research lines
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Definition of a location problem

- spatial resource allocation problem
- concepts of **facilities** serving **demand**
- spatial topology

- spatially dependent objective
- typical criteria:
  - minimizing average travel time or distance
  - minimizing average response time
  - minimizing a cost function of travel or response time
  - minimizing max travel time
  - maximizing min travel time
Definition of a location problem


A couple of famous examples of location problems on networks

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**Terminology and notation**

- planar network \( \mathcal{N} = (V, E) \)
- no multiedges, no loops, rectifiable edges
- for notational simplicity, identification \( \mathcal{N} \) with the set of points contained in any edge \( e \in E \)
- the definition of the **distance** between any two points \( x, y \in \mathcal{N} \) as the length of any shortest path in \( \mathcal{N} \) joining \( x \) and \( y \)
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The $p$–center problem

**Definition**

Let $f$ be the function defined for $X \subset V$ by

$$f(X) = \max \{ f_i(d(v_i, X)) : i \in I \},$$

for any given nondecreasing functions $f_i, i \in I$.

The $p$–center problem is to find an absolute $p$–center $X^* = \{x_1^*, \ldots, x_p^*\}$ and the $p$–radius $r_p$ such that

$$r_p \equiv f(X^*) = \min \{ f(X) : |X| = p, X \subset V \}.$$

- interpretation of $f_i(d(X, v_i))$
  - loss incurred while travelling from the closest center to $v_i$
  - time to travel from $v_i$ to the nearest service center
- vertex restricted and continuous case - the difference
- usual form of the function $f_i(d(v_i, X))$: $w_i d(v_i, X) + a_i$
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Let $g(X) = \sum_{i \in I} w_i d(v_i, X)$ for $X \subset \mathcal{N}$.

The $p$–median problem is to find a set $X^*$ of $p$ points for which $g(X^*) = \min\{g(X) : |X| = p, X \subset \mathcal{N}\}$.

- **Absolute** $p$–median of $\mathcal{N}$: any set $X^*$ of $p$ points minimizing $g$. Hakimi (1964, 1965): there exists an absolute $p$–median consisting entirely of vertices → the distinction between the vertex restricted and unrestricted versions is insignificant.

- Real life $p$–median problems:
  - locating plants/warehouses to serve other plants/warehouses or market areas
  - example of public sector location model
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- Hakimi (1983): customers deterministically choose the facility (the nearest one)
- Huff (1963): customers divide their patronage probabilistically
- Gravity-based formula: the probability that a consumer patronizes a shopping center is proportional to the attractiveness of the center and inversely proportional to a power of the distance to it
- Both demand and facilities locations can be at nodes and also edges
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Huff location model with continuous demand

- \( m \) facilities already located at points \( y_1, \ldots, y_m \) on the network
- customers located all over the network \( \mathcal{N} \), thus with \( x \) on \( \mathcal{N} \) is associated the demand density \( w(x) \) with the properties:
  - \( w_e(x) \geq 0 \), on each edge \( e \in E \), and
  - \( \sum_{e \in E} \int_e w_e(x) \, dx = 1 \)

- The goal is to locate new \( p \) facilities which will respond to the customers’ demands in such a way that the captured demand is maximal.
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- \( a_{s_i} \): attractiveness of store at \( s_i \)
- \( d(x, s_i) \): distance from the customers located at the arbitrary point \( x \) to the store at \( s_i \) on \( N \)
- \( F(d(x, s_i)) \): distance deterrence function of the customers \( x \) from the store at \( s_i \) (monotonically decreasing function with respect to \( d(x, s_i) \))
  - by Huff’s original model
  \[
  F(d(x, s_i)) = d(x, s_i)^{-\lambda}, \quad \lambda > 0
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- \( P(x, s_i) \): probability of consumer at \( x \) choosing store at \( s_i \) among the \( m + p \) stores
- the network Huff model is
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Introduction

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<table>
<thead>
<tr>
<th>Introduction</th>
<th>Problem definition</th>
<th>Algorithms and results</th>
<th>Conclusions and future research lines</th>
</tr>
</thead>
<tbody>
<tr>
<td>Huff location model with discrete demand</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Huff location model with discrete demand

\( p - \text{facility case} \)

- customers are located at the vertices \( v_1, \ldots, v_n, \ n = |V|, \) of the network
- the probability \( P(v_k, s_i) \) of customer at \( v_k \) choosing facility at \( s_i \) among the \( m + p \) facilities
  \[
  P(v_k, s_i) = \frac{a_s d(v_k, s_i)^{-\lambda}}{\sum_j a_s d(v_k, s_j)^{-\lambda}}
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- the demand \( D(v_k, s_i) \) of the customer \( v_k \) that facility at \( s_i \) captures
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  D(v_k, s_i) = \frac{a_s d(v_k, s_i)^{-\lambda}}{\sum_j a_s d(v_k, s_j)^{-\lambda}} w_k
  \]
- the total demand captured by the new facility located at \( z_i \)
  \[
  D(z_i) = \sum_{k=1}^n \frac{a_{z_i} d(v_k, z_i)^{-\lambda}}{\sum_j a_s d(v_k, s_j)^{-\lambda}} w_k
  \]
customers are located at the vertices \( v_1, \ldots, v_n, n = |V| \), of the network

the probability \( P(v_k, s_i) \) of customer at \( v_k \) choosing facility at \( s_i \) among the \( m + p \) facilities

\[
P(v_k, s_i) = \frac{a_{s_i}d(v_k, s_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}}
\]

the demand \( D(v_k, s_i) \) of the customer \( v_k \) that facility at \( s_i \) captures

\[
D(v_k, s_i) = \frac{a_{s_i}d(v_k, s_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} w_k
\]

the total demand captured by the new facility located at \( z_i \)

\[
D(z_i) = \sum_{k=1}^{n} \frac{a_{z_i}d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} w_k
\]
Huff location model with discrete demand

$p$-facility case

- Customers are located at the vertices $v_1, \ldots, v_n$, $n = |V|$, of the network.

- The probability $P(v_k, s_i)$ of customer at $v_k$ choosing facility at $s_i$ among the $m + p$ facilities
  \[ P(v_k, s_i) = \frac{a_{s_i}d(v_k, s_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} \]

- The demand $D(v_k, s_i)$ of the customer $v_k$ that facility at $s_i$ captures
  \[ D(v_k, s_i) = \frac{a_{s_i}d(v_k, s_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} w_k \]

- The total demand captured by the new facility located at $z_i$
  \[ D(z_i) = \sum_{k=1}^{n} \frac{a_{z_i}d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} w_k \]
Huff location model with discrete demand

$p$—facility case

- Customers are located at the vertices $v_1, \ldots, v_n$, $n = |V|$, of the network.
- The probability $P(v_k, s_i)$ of customer at $v_k$ choosing facility at $s_i$ among the $m + p$ facilities
  \[ P(v_k, s_i) = \frac{a_{s_i}d(v_k, s_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} \]
- The demand $D(v_k, s_i)$ of the customer $v_k$ that facility at $s_i$ captures
  \[ D(v_k, s_i) = \frac{a_{s_i}d(v_k, s_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} w_k \]
- The total demand captured by the new facility located at $z_i$
  \[ D(z_i) = \sum_{k=1}^{n} \frac{a_{z_i}d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} w_k \]
Huff location model with discrete demand

**p−facility case**

- The objective function
  \[
  D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i}d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} W_k
  \]

- Finally, the problem of the competitive Huff location model with discrete demand
  \[
  \max_{z_1, \ldots, z_p \in N} D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i}d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}} W_k
  \]

- Under the assumption of differentiability of the function \( D \) at point \((z_1, \ldots, z_p) \in N^p\), the optimality condition
  \[
  a_{z_l} \sum_{k=1}^{n} \left( \frac{d(v_k, z_l)^{-\lambda-1}d'(v_k, z_l)}{\left(\sum_{s_j} a_{s_j}d(v_k, s_j)^{-\lambda}\right)^2} \sum_{i=1}^{m} a_{y_i}d(v_k, y_i)^{-\lambda} \right) w_k = 0, \]
  for \( l \in \{1, \ldots, p\} \)
Huff location model with discrete demand

$p-$facility case

- the objective function
  \[ D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i} d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j} d(v_k, s_j)^{-\lambda}} W_k \]

- finally, the problem of the competitive Huff location model with discrete demand
  \[ \max_{z_1, \ldots, z_p \in \mathcal{N}} D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i} d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j} d(v_k, s_j)^{-\lambda}} W_k \]

- under the assumption of differentiability of the function $D$ at point $(z_1, \ldots, z_p) \in \mathcal{N}^p$, the optimality condition
  \[ a_{z_l} \sum_{k=1}^{n} \left( \frac{d(v_k, z_l)^{-\lambda-1} d'(v_k, z_l)}{\left( \sum_{s_j} a_{s_j} d(v_k, s_j)^{-\lambda} \right)^2} \right) \sum_{i=1}^{m} a_{y_i} d(v_k, y_i)^{-\lambda} W_k = 0, \]
  for \( l \in \{1, \ldots, p\} \)
Huff location model with discrete demand

$p$–facility case

- the objective function
  \[ D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i} d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j} d(v_k, s_j)^{-\lambda}} w_k \]
- finally, the problem of the competitive Huff location model with discrete demand
  \[ \max_{z_1, \ldots, z_p \in \mathcal{N}} D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i} d(v_k, z_i)^{-\lambda}}{\sum_{s_j} a_{s_j} d(v_k, s_j)^{-\lambda}} w_k \]
- under the assumption of differentiability of the function $D$ at point $(z_1, \ldots, z_p) \in \mathcal{N}^p$, the optimality condition
  \[ a_{z_l} \sum_{k=1}^{n} \left( \frac{d(v_k, z_l)^{-\lambda-1} d'(v_k, z_l)}{\left( \sum_{s_j} a_{s_j} d(v_k, s_j)^{-\lambda} \right)^2} \sum_{i=1}^{m} a_{y_i} d(v_k, y_i)^{-\lambda} \right) w_k = 0, \]
  for $l \in \{1, \ldots, p\}$
Huff location model with discrete demand

1—facility case

- \( p = 1 \), i.e. locating a single new facility \( z_1 \) on a general network
- the objective function of the 1—facility competitive Huff location model with discrete demand
  \[ D(z_1) = a_z z_1 \sum_{k=1}^{n} \frac{d(v_k, z_1)^{-\lambda}}{\sum_{s_j} a_{s_j} d(v_k, s_j)^{-\lambda}} w_k \]
- under the assumption of differentiability of the function \( D \) at point \( z_1 \), the optimality condition
  \[ a_z z_1 \sum_{k=1}^{n} \frac{B_k d(v_k, z_1)^{-\lambda-1} d'(v_k, z_1)}{(B_k + a_z z_1 d(v_k, z_1)^{-\lambda})^2} w_k = 0 \]
$\rho = 1$, i.e. locating a single new facility $z_1$ on a general network

The objective function of the 1–facility competitive Huff location model with discrete demand

$$D(z_1) = a_{z_1} \sum_{k=1}^{n} \frac{d(v_k,z_1)^{-\lambda}}{\sum_{s_j} a_{s_j} d(v_k,s_j)^{-\lambda}} w_k$$

under the assumption of differentiability of the function $D$ at point $z_1$, the optimality condition

$$a_{z_1} \sum_{k=1}^{n} \frac{B_k d(v_k,z_1)^{-\lambda-1} d'(v_k,z_1)}{(B_k + a_{z_1} d(v_k,z_1)^{-\lambda})^2} w_k = 0$$
Huff location model with discrete demand

1–facility case

- $p = 1$, i.e. locating a single new facility $z_1$ on a general network
- the objective function of the 1–facility competitive Huff location model with discrete demand
  $$D(z_1) = a_{z_1} \sum_{k=1}^{n} \frac{d(v_k,z_1)^{-\lambda}}{\sum_{s_j} a_{s_j} d(v_k,s_j)^{-\lambda}} w_k$$
- under the assumption of differentiability of the function $D$ at point $z_1$, the optimality condition
  $$a_{z_1} \sum_{k=1}^{n} \frac{B_k d(v_k,z_1)^{-\lambda-1} d'(v_k,z_1)}{(B_k + a_{z_1} d(v_k,z_1)^{-\lambda})^2} w_k = 0$$
p–facility case on a segment

- The network is degenerated \( S = (N, E) \) into a segment and the distance \( d(x, y) \equiv |x - y| \) and \( \lambda = 2 \) are chosen; the objective function
  \[
  D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i}(v_k - z_i)^{-2}}{\sum_{s_j} a_{s_j}(v_k - s_j)^{-2}} w_k
  \]

- The optimality condition
  \[
  a_{z_l} \sum_{k=1}^{n} \left( \frac{(v_k - z_l)^{-3}}{\left( \sum_{s_j} a_{s_j}(v_k - s_j)^{-2} \right)^2} \sum_{i=1}^{m} a_{y_i}(v_k - y_i)^{-2} \right) w_k = 0, \quad l \in \{1, \ldots, p\}.
  \]
the network is degenerated $S = (N, E)$ into a segment and the distance $d(x, y) \equiv |x - y|$ and $\lambda = 2$ are chosen; the objective function

$$D(z_1, \ldots, z_p) = \sum_{i=1}^{p} \sum_{k=1}^{n} \frac{a_{z_i}(v_k-z_i)^{-2}}{\sum_{s_j} a_{s_j}(v_k-s_j)^{-2}} w_k$$

the optimality condition

$$a_{z_l} \sum_{k=1}^{n} \left( \frac{(v_k-z_l)^{-3}}{\left(\sum_{s_j} a_{s_j}(v_k-s_j)^{-2}\right)^2} \sum_{i=1}^{m} a_{y_i}(v_k - y_i)^{-2} \right) w_k = 0,$$

$l \in \{1, \ldots, p\}$
Outline

1. Introduction
   - Definition of a location problem
   - A couple of famous examples of location problems on networks
   - Location problems in a competitive environment

2. Problem definition
   - Huff location model with continuous demand
   - Huff location model with discrete demand

3. Algorithms and results
   - VNS metaheuristic
   - Computational results

4. Conclusions and future research lines
Variable Neighborhood Search - metaheuristic for solving global optimization problems global min$_{x \in X} f(x)$

avoiding **entrapments in local minima** gives near-optimal solutions

concept of VNS - a very simple one

GLOB - software for minimization of a **continuous** function subject to **box** constraints

built-in VNS metaheuristic used
Repeat until the predefined stopping criterion is met:

- (1) Set $k \leftarrow 1$
- (2) Until $k > k_{\text{max}}$ repeat the following steps:
  - (a) *Shaking*: Generate a point $x'$ at random from $N_k(x)$.
  - (b) *Local search*: Apply some local search method with $x'$ as the initial solution; denote by $x''$ the so obtained local minimum.
  - (c) *Move or not*: If $x''$ is better than the incumbent, move there ($x \leftarrow x''$) and set $k \leftarrow 1$; otherwise set $k \leftarrow k + 1$. 

Outline

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   - Computational results

4. Conclusions and future research lines
Design of experiments

- location problems with discrete demand on a segment
- \( d(x, y) \equiv |x - y|, \lambda = 2 \)
- number of customers: 2, 50, 100, 500 and 1000
- customers and demand, attractiveness of the facilities and locations of the existing ones - all uniformly chosen
## Results

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<th>5</th>
<th>10</th>
<th>20</th>
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**Table:** The total demand captured by new facilities in case of 2 customers
## Results

<table>
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<tr>
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</table>

**Table:** The total demand captured by new facilities in case of 50 customers
Results

Table: The total demand captured by new facilities in case of 100 customers

<table>
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</thead>
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<td>57.27996</td>
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</table>
## Results

<table>
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<td>53.71175</td>
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</table>

**Table:** The total demand captured by new facilities in case of 500 customers
### Results

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</table>

**Table:** The total demand captured by new facilities in case of 1000 customers
Conclusions and future research lines

- first step: competitive Huff location model with discrete demand on a segment
- our ultimate goal: competitive Huff location model with continuous demand on networks
Thank you!