Divisor class groups of singular varieties

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Sevilla, 2017
Joint papers with Lê Dũng Tráng (selection):


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Introduction

$X$ complex algebraic variety (reduced, of pure dimension)
Weil divisor on $X$: $\sum n_j D_j$, $n_j \in \mathbb{Z}$, $D_j$ irreducible hypersurface in $X$
Principal divisor: of the form $\sum (\text{ord}_D f) D$, with $f$ rational and invertible on $X$.
Frequent assumption: $\text{codim}_X \text{Sing}(X) \geq 2$, then notion $\text{ord}_D f$ clear.
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In general one can reduce to this case using normalization $\pi : \hat{X} \to X$:
$\text{ord}_D f := \sum (\text{ord}_{\hat{D}_j} \hat{f})\deg(\hat{D}_j \to D)$, where $\hat{D}_1, \ldots$ are the irreducible components of $\pi^{-1}(D)$. See Fulton, Intersection theory.
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$\text{Cl}(X) := \{\text{Weil divisors}\}/\{\text{principal divisors}\}$: Weil divisor class group.
Aim: Lefschetz (hyperplane section) theorem for $\text{Cl}(X)$. 
Smooth case

If $X$ is smooth: $Cl(X) \simeq Pic(X) \simeq H^1(X, \mathcal{O}_X^*)$. 

Theorem $f$: Morphism between smooth varieties such that $H^j(X; Z) \simeq H^j(Y; Z)$, $j = 1, 2$, then $Pic(X) \simeq Pic(Y)$.

Proof: Case $X, Y$ compact: switch to analytic category (GAGA), exponential sequence: $H^1(X; Z) \to H^1(X, \mathcal{O}_X) \to Pic(X) \to H^2(X; Z) \to H^2(X, \mathcal{O}_X) \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \to H^1(Y; Z) \to H^1(Y, \mathcal{O}_Y) \to Pic(Y) \to H^2(Y; Z) \to H^2(Y, \mathcal{O}_Y)$.

By assumption, first and fourth vertical arrow bijective. So by Hodge theory, second and fifth arrow, too. Now apply five lemma. General case: [H-L 2005], using mixed Hodge structures.
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Let $f : Y \to X$ be a morphism between smooth varieties such that $H^j(X; \mathbb{Z}) \simeq H^j(Y; \mathbb{Z})$, $j = 1, 2$, then $Pic(X) \simeq Pic(Y)$. 

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a) $H^j(Z, f^{-1}(U); \mathbb{Z}) = 0, j \leq n - 1,$
A Lefschetz theorem for integral cohomology

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Special case: \( f \) inclusion: \( Z \) quasi-projective.
Lefschetz theorem for Weil divisor class groups

**Theorem**

Let $X \subset \mathbb{P}_N$ be a quasi-projective variety, $\dim X \geq 4$, $H$ general hyperplane in $\mathbb{P}_N$, $Y := X \cap H$. Then $Cl(X) \simeq Cl(Y)$.
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Proof: The case $\text{codim}_X \text{Sing}(X) \geq 2$ has already been treated in [H-Lê, 2005]: Then $\text{Sing}(Y) = (\text{Sing}(X)) \cap H$ has codimension $\geq 2$ in $Y$, too.

By the two theorems before, $\text{Cl}(X \setminus \text{Sing}(X)) \simeq \text{Cl}(Y \setminus \text{Sing}(Y))$, so $\text{Cl}(X) \simeq \text{Cl}(X \setminus \text{Sing}(X)) \simeq \text{Cl}(Y \setminus \text{Sing}(Y)) \simeq \text{Cl}(Y)$. 
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**General case:** Let \( \pi : \hat{X} \to X \) be the normalization.

After removing some subspace of codimension \( \geq 2 \) (which has no influence on \( Cl(X), Cl(Y) \)) we may assume:

\( \hat{X} \) smooth, \( D := \text{Sing}(X) \) smooth of codimension 1, 

\( \pi : \pi^{-1}(D) \to D \) unramified covering.

\( D = D_1 \cup \ldots \cup D_r \): decomposition into connected components, 

\( \pi^{-1}(D_j) = \hat{D}_{j1} \cup \ldots \) likewise.
By Lefschetz for integral homology:

a) \( H^0(D; \mathbb{Z}) \simeq H^0(D \cap H; \mathbb{Z}) \), so \( D^*_j := D_j \cap H \) is connected \( \neq \emptyset \),

b) \( H^0(\pi^{-1}(D_j); \mathbb{Z}) \simeq H^0(\pi^{-1}(D_j \cap H); \mathbb{Z}) \), i.e. \( \hat{D}^*_j := \hat{D}_{jk} \cap H \) is connected \( \neq \emptyset \).

By the two theorems before (note: \( X \setminus D \) and \( \hat{X} \) are smooth):

c) \( Cl(X \setminus D) \simeq Cl(Y \setminus D) \),

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Because of a):

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Of course, the first vertical arrow is bijective; by c) the third one is bijective, too.

Need: \( \text{im } \phi \to \text{im } \psi \) injective.

Let \( \sum n_j D_j \) be mapped to a principal divisor \( \text{div } f \) on \( Y \), \( f \) rational on \( Y \).

We must show: \( \sum n_j D_j \) is a principal divisor (on \( X \)), too.

Now \( \text{div } f = \text{push-forward of } \text{div } \hat{f} \), \( \hat{f} \) corresponding rational function on \( \hat{Y} \).

By d): \( \text{div } \hat{f} = \text{image of } \text{a principal divisor } \text{div } \hat{g} \) on \( \hat{g} \).

Then \( \sum n_j D_j = \text{div } g \) principal divisor, \( g \) corresponding rational function on \( X \).
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Now \(div f = \) push-forward of \(div \hat{f}\), \(\hat{f}\) = corresponding rational function on \(\hat{Y}\). By d): \(div \hat{f} = \) image of a principal divisor \(div \hat{g}\) on \(\hat{g}\).

Then \(\sum n_j D_j = div g\) principal divisor, \(g\) corresponding rational function on \(X\).
We skipped the question how to define the pull-back $\text{Cl}(X) \to \text{Cl}(Y)$. In the case of the Picard group we have always a pull-back, in the case of $\text{Cl}$ one needs a condition on the mapping. Here: "Gysin map" in the sense of Fulton, Intersection theory. Also for flat mappings pull-back defined, one can use similar method as above to show isomorphy of $\text{Cl}$.
Lefschetz theorem for Cartier divisor class groups

In the singular case Weil and Cartier divisors are usually different. For quasi-projective varieties: Cartier divisor class group = Picard group.

A Lefschetz theorem for the Picard group has been derived in [H-Lê, 2010], it involves depth conditions:

Theorem
Let $X \subset \mathbb{P}^N$ be a quasi-projective variety, $\dim X \geq 4$, $H$ general hyperplane in $\mathbb{P}^N$, $Y := X \cap H$. Assume:

- $\text{depth} \, \text{Sing}(X) \geq 3$,
- $H^3(X, X \{x\}; \mathbb{Z}) = 0$ for all $x \in X$.

Then $\text{Pic}(X) \cong \text{Pic}(Y)$.

Note that the assumption implies that $X$ and $Y$ are normal, in this case Lefschetz for Weil divisor classes was easier to prove.

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Use of neighbourhoods

In Lefschetz for Weil divisor classes: better result if we replace $H$ by a good neighbourhood $U$ of $H$ - in transcendental topology.
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This changes in the case of divisor classes. Here we obtain weaker hypotheses with $U$. Cf. to Grothendieck, SGA2.
Theorem

([H, 2008]) Suppose that $X$ is projective of dimension $\geq 3$. Then $\text{Cl}(X) \cong \text{Cl}(X \cap U)$.

But it seems hopeless to compare $U \cap X$ and $H \cap X$ directly.
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(loc. cit.) Suppose that $X$ is projective, $\text{depth } \mathcal{O}_{X \setminus H} \geq 3$, $H^3(X, X \setminus \{x\}; \mathbb{Z}) = 0$ for all $x \in X \setminus H$. Then $\text{Pic}(X) \cong \text{Pic}(X \cap U)$.

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Chow groups of an algebraic variety $X$: $A_k(X)$. 
Put $A^k(X) := A_{n-k}(X)$, $n := \dim X$ - even if $X$ is singular. 
$A^0(X) \cong \mathbb{Z}^r$ if $X$ has $r$ irreducible components. 
$A^1(X) = Cl(X)$. 

Theorem: Suppose that $X$ is a projective variety, $H$ general hyperplane, $\dim X \geq 2k + 2$. Then $A^k(X) \cong A^k(Y)$ if $k = 0, 1$. 
Proof: $k = 0$: Apply Lefschetz for integral homology to $X \setminus \text{Sing}(X)$. 
$k = 1$: see Lefschetz for Weil divisor classes.
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**Proof:** $k = 0$: Apply Lefschetz for integral homology to $X \setminus Sing(X)$.
$k = 1$: see Lefschetz for Weil divisor classes.
What happens if $k > 1$?
Special case: Let $Z$ be a smooth hypersurface in $\mathbb{P}_n$. We can write $Z \simeq X \cap H$, $X \simeq \mathbb{P}_n$ projective. If the preceding result generalized to any $k$ we would have $A^k(Z) \simeq \mathbb{Z}$ if $n \geq 2k + 2$. Cf. Hartshorne’s question.
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Too naive approach for surjectivity: Induction on $k$. Induction step from $k - 1$ to $k$:

Let $C$ be an irreducible subvariety of $Y$ of codimension $k$. Then there is a homogeneous polynomial $g$ such that $g|C = 0$, $g|Y \neq 0$. Put $Z := X \cap \{g = 0\}$. Then $Z$ and $Z \cap Y$ are divisors on $X$ resp. $Y$. If we could ignore the hypothesis that $H$ is general (which is not possible!) we would obtain, by induction hypothesis, that $A^{k-1}(Z) \to A^{k-1}(Z \cap Y)$ is surjective. Now $C$ represents an element of $A^{k-1}(Z \cap Y)$, so we would find a linear combination of irreducible subspaces of codimension $k - 1$ in $Z$, hence of codimension $k$ in $X$ whose class is mapped to the one of $C$. It represents an inverse image of the class of $C$ under $A^k(X) \to A^k(Y)$. 
Easier if we work with $U$:

**Theorem**

(see [H, 2010]) Suppose that $X$ is a projective variety, $U$ a suitable neighbourhood of a hyperplane $H$, $\dim X \geq k + 2$. Then $A^k(X) \rightarrow A^k(X \cap U)$ is bijective.
Another application

In the proof of we needed that the restriction is well-defined. This is not only true for Gysin homomorphisms, here another example where a similar approach works:

Let $w_1, \ldots, w_m > 0$ be integers which are relatively prime, $w := (w_1, \ldots, w_m)$ and $\mathbb{P}_w := (\mathbb{C}^m \setminus \{0\})/\mathbb{C}^*$ the corresponding weighted projective space.

Let $V$ be a purely $n$-dimensional subvariety of $\mathbb{P}_w$, $n \geq 1$. It is of the form $X \setminus \{0\}/\mathbb{C}^*$ where $X \subset \mathbb{C}^m$ is defined by weighted homogeneous polynomials. We assume that no irreducible component of $X$ is contained in a coordinate hyperplane.

Put $\tilde{w} := (1, w_1, \ldots, w_m)$. Then $\mathbb{P}_{\tilde{w}}$ is a compactification of $\mathbb{C}^m$.

Let $\bar{X}$ be the closure of $X$ in $\mathbb{P}_{\tilde{w}}$. Put $X_\infty := \bar{X} \setminus X$. 
Proposition

a) \( Cl(V) \cong Cl(\tilde{X}) \).
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b) Suppose that \( V \) is irreducible: Then there is an exact sequence
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hence \( Cl(V)_\mathbb{Q} \cong Cl(X) \oplus \mathbb{Q} \).
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Proof: a) According to Fulton, Intersection theory, there is a pull-back for flat mappings, so the projection $\pi: \bar{X} \setminus \{0\} \to V$ induces $\text{Cl}(V) \to \text{Cl}(\bar{X} \setminus \{0\})$.

Note that $\pi|_{X_{\infty}}$ is bijective, so the composed mapping $\text{Cl}(V) \to \text{Cl}(\bar{X} \setminus \{0\}) \to \text{Cl}(X_{\infty})$ is bijective.

Hence $\text{Cl}(V) \to \text{Cl}(\bar{X} \setminus \{0\})$ must be injective.

Let $D$ be the divisor in $V$ obtained by intersecting with all coordinate hyperplanes. Then we have a commutative diagram
\[ \mathbb{Z}^m \rightarrow Cl(V) \rightarrow Cl(V \setminus D) \rightarrow 0 \]
\[ \mathbb{Z}^m \rightarrow Cl(\tilde{X} \setminus \{0\}) \rightarrow Cl(\pi^{-1}(V \setminus D)) \rightarrow 0 \]

Now the last vertical arrow is induced by the projection of a line bundle onto its base, hence bijective by [Fu].
The first vertical arrow is bijective, of course. So the second one is surjective, and we know already the injectivity.
Finally, \( Cl(\tilde{X}) \cong Cl(\tilde{X} \setminus \{0\}) \) because \( n + 1 \geq 2 \).

b) Note that \( X_\infty \cong V \) is irreducible, hence there is an exact sequence \( \mathbb{Z} \rightarrow Cl(\tilde{X}) \rightarrow Cl(X) \rightarrow 0 \). And the first mapping must be injective.