Bifurcation Theory and its applications in PDEs and Mathematical Biology
Lecture 1: Abstract Bifurcation Theory of Stationary Problems

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Where is Virginia ?

Where is *College of William and Mary* ?

1. College of William and Mary is in Williamsburg, Virginia, USA, about 200 km south of Washington DC.
2. College of William and Mary is 2nd oldest university in USA, founded in 1693 (1st settlement in Virginia was established in 1607). William/Mary were the king/queen of UK at that time.
3. Williamsburg was the capital of Virginia before America revolution, and George Washington lived there for some time when he served the senator of Virginia State. Thomas Jefferson was a law student in William and Mary.
4. College of William and Mary is ranked number 10 among all public universities and number 38 among all national universities by US News. (there are more than 4000 universities in USA)

Oldest University Academic Building in US

Beautiful Campus

Research areas

- Nonlinear Partial Differential Equations (Elliptic and Parabolic Type).
- Reaction-diffusion models: diffusion, advection, and nonlocal effect.
- Applied Nonlinear Analysis; Bifurcation Theory; Infinite Dimensional Dynamical Systems.
- Delay Differential Equations; Ordinary Differential Equations.
- Mathematical Biology; Natural Resource Modeling; Spatiotemporal Pattern Formation.
- Other Applied Mathematics (Calculus of Variation, Celestial Mechanics, Mathematical Physics, etc.)

Course Plan

My webpage: `http://jxshix.people.wm.edu/`
My publication webpage: `http://jxshix.people.wm.edu/publication.html`

1. Lecture 1 (October 16): Abstract Bifurcation Theory for Stationary Problems
2. Lecture 2 (October 17): Turing Type Bifurcations
3. Lecture 3 (October 17): Global Bifurcation and Exact Multiplicity
4. Lecture 4 (October 18): Hopf Bifurcations
5. Lecture 5 (October 19): Beyond Diffusion: advection, taxis, and nonlocal effect
6. Lecture 6 (October 20): Delay Differential Equations

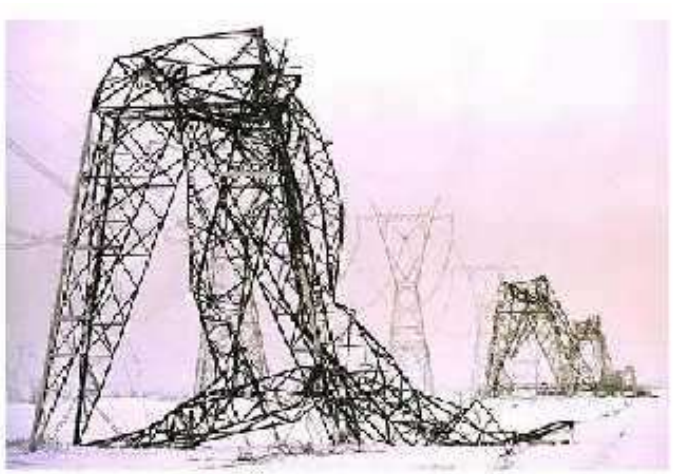
First Word

John von Neumann (1903-1957)
If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

Cliff Taubes (1954-)
http://w3.math.sinica.edu.tw/media/media.jsp?voln=371

"What is the most useful mathematics for you?"
"I think one is fundamental theorem of calculus, and another is maximum principle."
In my opinion, another two important things are: (i) Taylor expansion of a function; (ii) implicit function theorem.

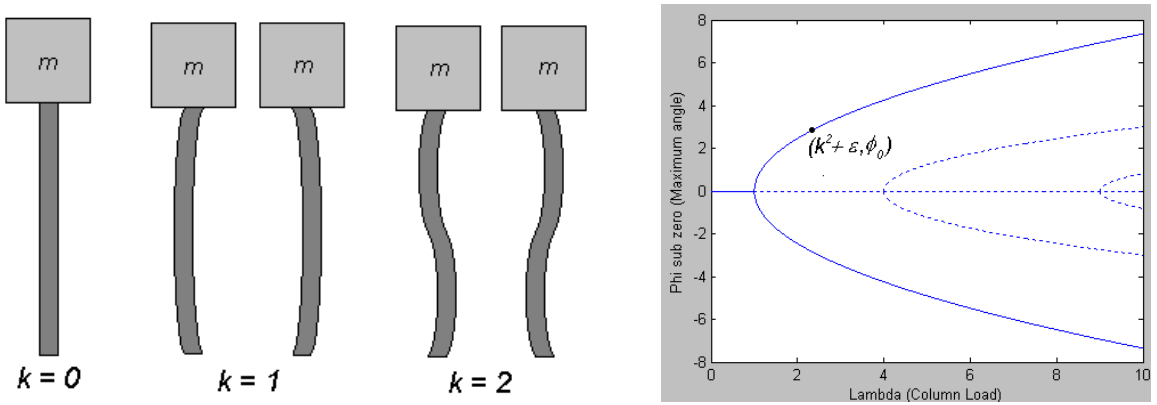
Earliest Bifurcation: Buckling



(Left: Buckling due to glaze icing; Right: Buckling of rails)

In engineering, buckling is a failure mode characterized by a sudden failure of a structural member subjected to high compressive stresses, where the actual compressive stresses at failure are greater than the ultimate compressive stresses that the material is capable of withstanding. This mode of failure is also described as failure due to elastic instability.

Euler Buckling Equation



(Left: buckling under different thrusts; Right: bifurcation diagram (λ, ϕ))

$\phi'' + \lambda \sin\phi = 0,\ x \in (0,\pi),\ \phi(0) = \phi(\pi) = 0$

$\phi(x)$ is the angle between the tangent to the column's axis and the y-axis, and λ is a parameter proportional to the thrust.

The word bifurcation is from the Latin *bifurcus*, which means two forks.

[Da Vinci, *Codex Madrid*, 1478-1519], [Euler, 1757], Galileo, Bernoulli

Da Vinci's sketching showing buckling

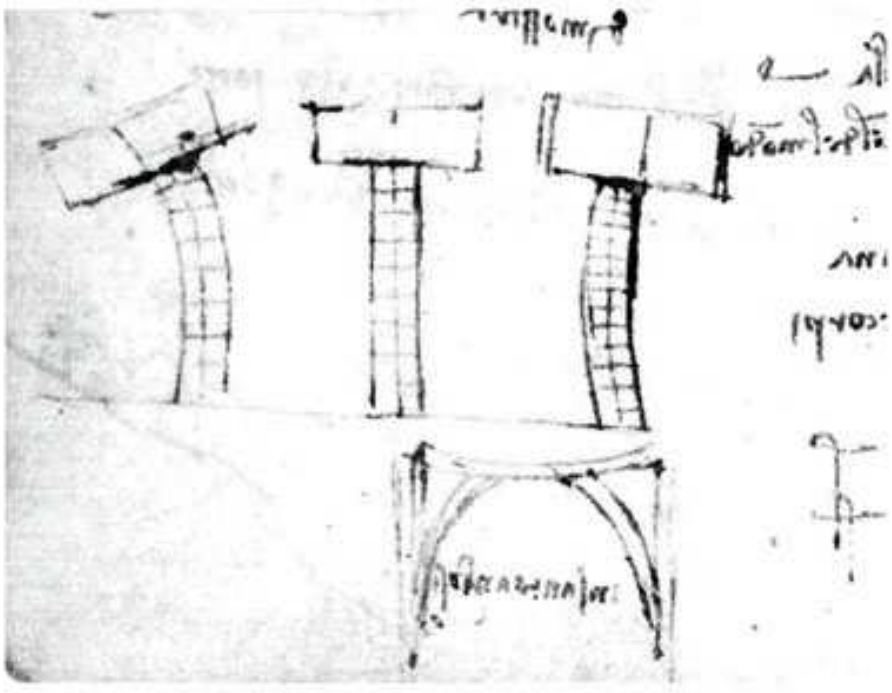
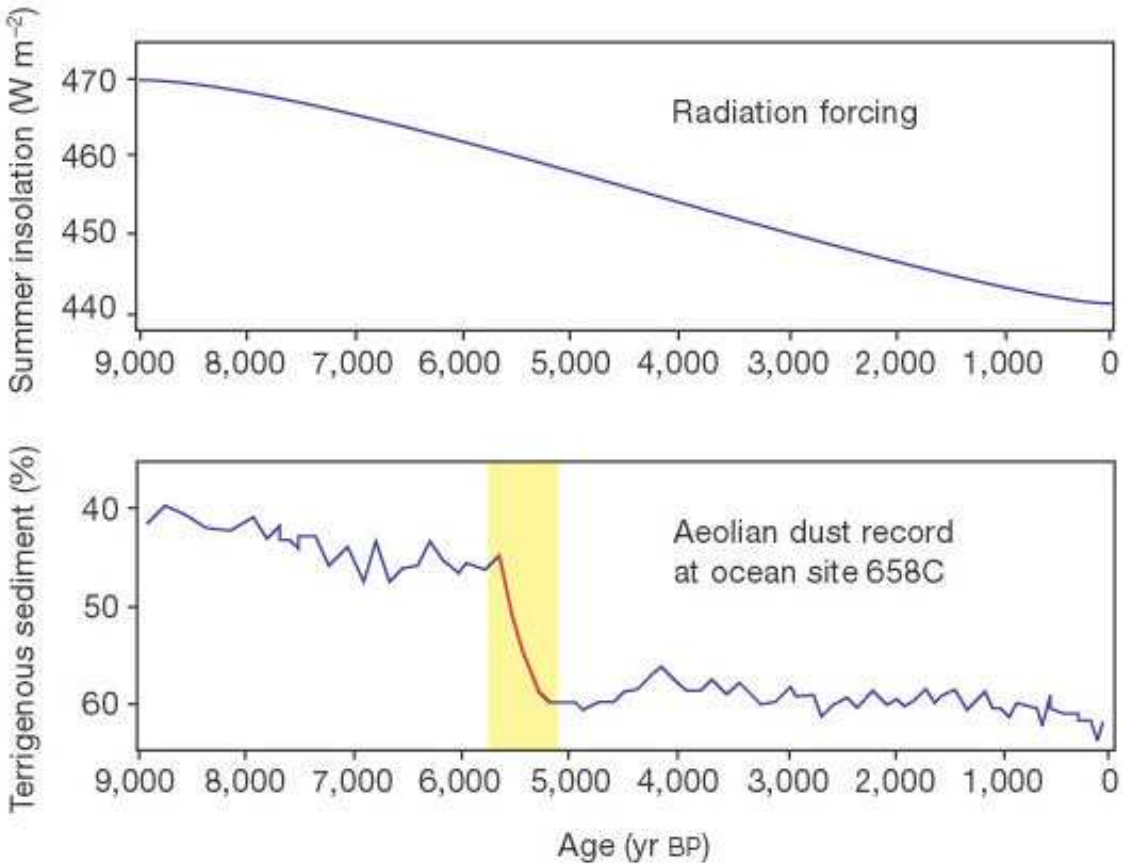


Figure 4 Leonardo da Vinci: drawing to illustrate the bending of a column under a vertical weight [from *Codex Madrid I* [9], page 117].

Formation of Sahara Desert



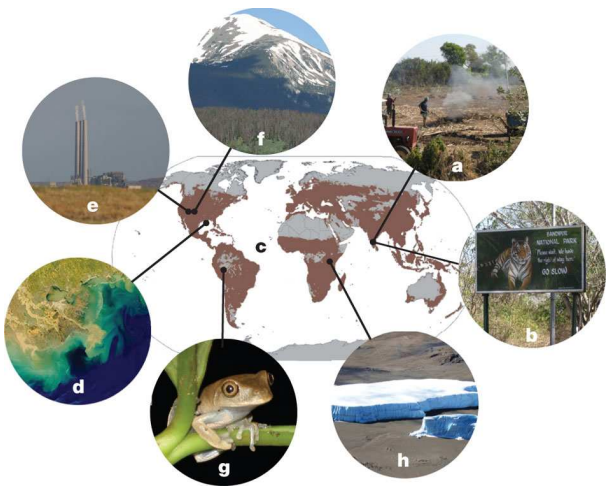
About 5000 years before present (yr bp), the change in solar radiation triggered an abrupt shift in climate and vegetation cover over the Sahara, as reflected in the contribution of terrigenous (land-eroded) dust to oceanic sediment at a sample site near the African coast (lower panel).

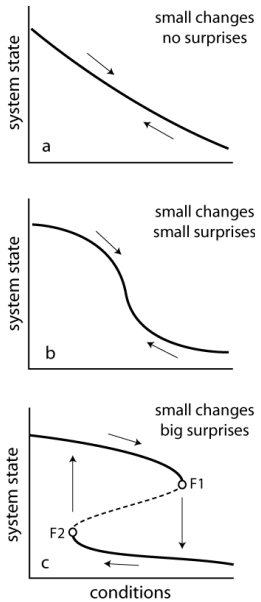
From: Nature 413, 591-596 (11 October 2001) doi:10.1038/35098000

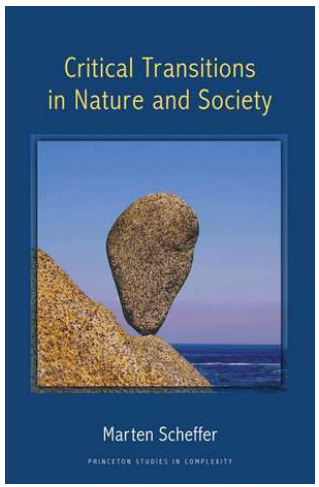
Bifurcations in Nature: global warming, critical transition

A sudden critical transition can bring catastrophic change to the earth ecosystem, thus it is important to predict and prevent it. `www.early-warning-signals.org`

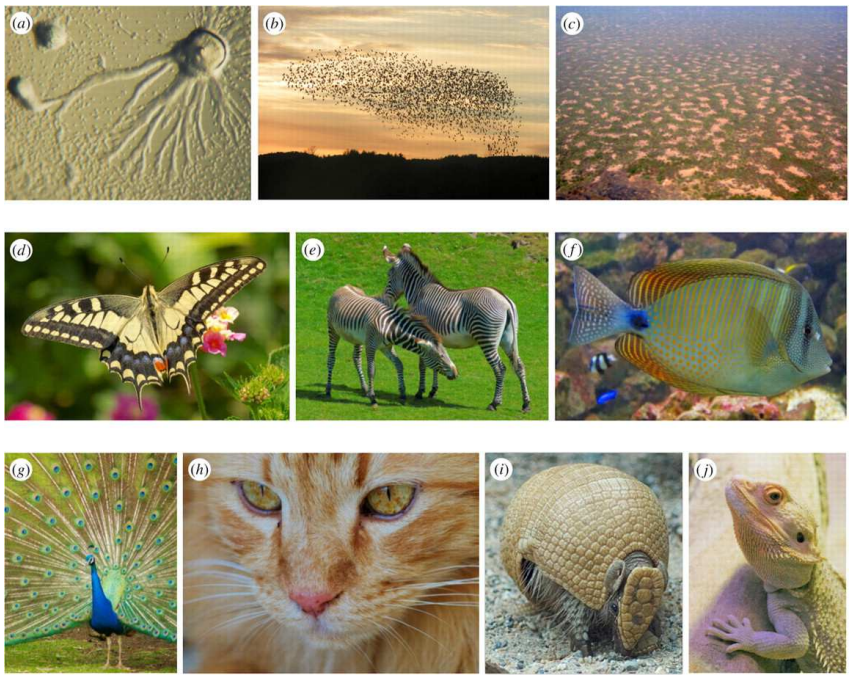
Critical Transitions in Nature and Society, by Marten Scheffer
`www.sparcs-center.org/about-us/staff-contacts/marten-scheffer.html`

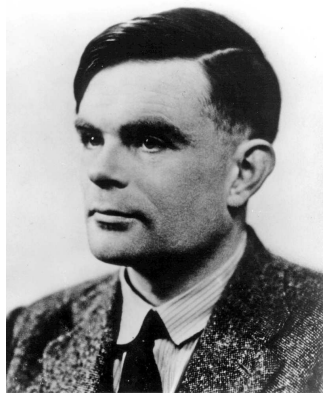






Pattern formation in biology





[Turing, 1952] The Chemical Basis of Morphogenesis. *Phil.Trans.Royal Soc. London.B*

Reaction-diffusion system (R-D): $u_t = d_1 \Delta u + f(u, v)$, $v_t = d_2 \Delta v + g(u, v)$
A spatially homogenous steady state is stable w.r.t. homogenous perturbation, but is unstable w.r.t. inhomogeneous perturbation. Hence some stable non-constant non-equilibrium state (patterns) exist to show the complex spatial-temporal structure, as a result of symmetry breaking bifurcation.

Figure (left) from: Towards an integrated experimentaltheoretical approach for assessing the mechanistic basis of hair and feather morphogenesis, by K. J. Painter et.al. Interface Focus. 2012 Aug 6;2(4):433-50

Banach spaces

1. **Metric space**: a pair (M, d) where M is a set and $d : M \times M \to \mathbb{R}$ is a metric which satisfies for any $x, y, z \in M$: (i) $d(x, y) \geq 0$; (ii) $d(x, y) = 0$ if and only if $x = y$; (iii) $d(x, y) = d(y, x)$; and (iv) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.
2. **Complete**: A metric space (M, d) is *complete* if any Cauchy sequence $\{x_n\} \subset M$ has a limit in M.
3. **Normed vector space**: a pair $(V, ||\cdot||)$ where V is a linear vector space over real numbers and the norm $||\cdot|| : V \to \mathbb{R}$ is a function which satisfies for any $a \in \mathbb{R}$ and $x, y \in V$: (i) $||ax|| = |a| \cdot ||x||$; (ii) $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$ (the zero vector); and (iii) (triangle inequality) $||x + y|| \leq ||x|| + ||y||$. A normed vector space is a metric space with the metric $d(x, y) = ||x - y||$.
4. **Banach space**: a complete normed vector space.
5. **Linear operator**: Let X and Y be Banach spaces. A mapping $F : X \to Y$ is linear if (i) $F(ax) = aF(x)$ and (ii) $F(x + y) = F(x) + F(y)$ for any $a \in \mathbb{R}$ and $x, y \in X$.
6. **Nonlinear operator**: A mapping $F : X \to Y$ which is not linear.
7. **Dimension**: For $n \in \mathbb{N}$, a Banach space X is n-dimensional if there is a subset of X with n non-zero linear independent elements, but there is no such a subset with $n + 1$ elements.
8. Example: $\mathbb{R}^n$, $C(\overline{\Omega})$ (the set of continuous functions defined on $\overline{\Omega}$).

Nonlinear Functions and Derivatives in Banach Spaces

Equation: $F(\lambda, u) = 0$, $F : \mathbb{R} \times X \to Y$ (X and Y are Banach spaces)

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Partial derivatives: (Frechét derivative or Gâteaux derivative)

$$F_u(\lambda, u)[\phi] = \lim_{h \to 0} \frac{F(\lambda, u + h\phi) - F(\lambda, u)}{h}$$

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If $F(\lambda, u) = \Delta u + \lambda f(u)$, then
$F_u(\lambda, u)[\phi] = \Delta\phi + \lambda f'(u)\phi$, $F_\lambda(\lambda, u) = f(u)$, $F_{\lambda u}(\lambda, u) = f'(u)$.
$F_{\lambda\lambda}(\lambda, u) = 0$, $F_{uu}(\lambda, u)[\phi, \psi] = \lambda f'(u)\phi\psi$.

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$F_u(\lambda, u)$ is a linear mapping from X to Y
$\mathcal{N}(F_u(\lambda, u)) \subset X$ is the null space (the space of solutions of $F_u(\lambda, u)[\phi] = 0$)
$\mathcal{R}(F_u(\lambda, u)) \subset Y$ is the range space

Derivatives as linear mappings

$F_u(\lambda, u) : X \to Y$ is a linear mapping.
$F_\lambda(\lambda, u) : \mathbb{R} \to Y$ is a linear mapping. But we use $F_\lambda(\lambda, u)[1]$ as $F_\lambda(\lambda, u)$ since $F_\lambda(\lambda, u)[\tau]$ is $\tau F_\lambda(\lambda, u)[1]$. Hence we can understand that $F_\lambda(\lambda, u) \in Y$.

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$F_{uu}(\lambda, u) : X \times X \to Y$ is a bilinear mapping (linear in both variables). That is,
(i) $F_{uu}(\lambda, u)[\phi_1 + \phi_2, \psi] = F_{uu}(\lambda, u)[\phi_1, \psi] + F_{uu}(\lambda, u)[\phi_2, \psi]$;
$F_{uu}(\lambda, u)[\phi, \psi_1 + \psi_2] = F_{uu}(\lambda, u)[\phi, \psi_1 + \psi_2] + F_{uu}(\lambda, u)[\phi, \psi_1 + \psi_2]$.
(ii) $F_{uu}(\lambda, u)[a\phi, \psi] = aF_{uu}(\lambda, u)[\phi, \psi]$; $F_{uu}(\lambda, u)[\phi, b\psi] = bF_{uu}(\lambda, u)[\phi, \psi]$.

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Similarly, $F_{uuu}(\lambda, u) : X \times X \times X \to Y$ is a trilinear mapping.

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Finite dimensional case:
$F = F(\lambda, u) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$.
$F_u = (J_{ij} = \partial_j F_i)$, $n \times n$ Jacobian matrix
$F_{\lambda u} = (M_{ij} = \partial_{\lambda j} F_i)$, $n \times n$ matrix
$F_{uu} = (K_{ijk} = \partial_{jk} F_i)$, $n \times n \times n$ Hessian matroid

Linear operator: $F_u[(x_1, \cdots, x_n)] = \left(\sum_j J_{1j} x_j, \cdots, \sum_j J_{nj} x_j \right)$

Bilinear operator:

$$F_{uu}[(x_1, \cdots, x_n), (y_1, \cdots, y_n)] = \left(\sum_{j,k} K_{1jk} x_j y_k, \cdots, \sum_{j,k} K_{njk} x_j y_k \right).$$

Solving from implicit function

Implicit function: $f(x_1, x_2, x_3, \cdots, x_n) = 0$.
Can one solve x_i from an implicit function?

1. $f_1(\lambda, y) = 3\lambda + 4y - 5$. Then $3\lambda + 4y - 5 = 0$, $y = (5 - 3\lambda)/4$. (globally uniquely solvable)
2. $f_2(\lambda, y) = \lambda^2 + y^2 - 1$. Then $\lambda^2 + y^2 - 1 = 0$, $y = \pm\sqrt{1 - \lambda^2}$. (locally uniquely solvable if $\lambda \in (-1, 1)$, not uniquely solvable if near $\lambda = \pm 1$)
3. $f_3(\lambda, y) = \lambda - \sin ye^y$. Then $\lambda - \sin ye^y = 0$, linearization at $(\lambda, y) = (0, 0)$: $\lambda - y = 0$, then $y(\lambda) \approx \lambda$ near $\lambda = 0$. (uniquely solvable? cannot be explicitly solved)

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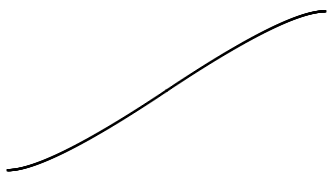
Suppose that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is smooth, and $f(\lambda_0, y_0) = 0$.
Implicit function theorem: If $f_y(\lambda_0, y_0) \neq 0$, then for λ near λ_0, $f(\lambda, y) = 0$ has a unique solution $y(\lambda)$ so that $f(\lambda, y(\lambda)) = 0$.
Submersion theorem (preimage theorem): If $\nabla f(\lambda_0, y_0) = (f_\lambda(\lambda_0, y_0), f_y(\lambda_0, y_0)) \neq (0, 0)$, then the set $\{(\lambda, y) : f(\lambda, y) = 0\}$ near (λ_0, y_0) is a curve.

Implicit Function Theorem: No Bifurcation

Theorem 1. Assume that X, Y are Banach spaces. Let $(\lambda_0, u_0) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y. Let $F(\lambda_0, u_0) = 0$ and $F_u(\lambda_0, u_0)$ is invertible ($F_u(\lambda_0, u_0)[\phi] = 0$ only has zero solution). Then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $(\lambda, u(\lambda))$, $u(\lambda) = u_0 + (\lambda - \lambda_0)w_0 + z(\lambda)$, where $w_0 = -[F_u(\lambda_0, u_0)]^{-1}(F_\lambda(\lambda_0, u_0))$ and $\lambda \mapsto z(\lambda) \in X$ is a continuously differentiable function near $s = 0$ with $z(0) = z'(0) = 0$.



Theorem 1a. (Finite dimensional) Suppose that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^k smooth mapping such that $f(\lambda_0, u_0) = 0$. Suppose that at $(\lambda_0, u_0) \in \mathbb{R}^m \times \mathbb{R}^n$, the matrix $f_u(\lambda_0, u_0) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exists a neighborhood A of λ_0 and a neighborhood B of u_0, such that for every $\lambda \in A$, there exists a unique $u = u(\lambda) \in B$ such that $f(\lambda, u) = 0$. The map $\lambda \mapsto u(\lambda)$ is C^k.

Implicit Function Theorem: more general version

Theorem 2. Let X, Y and Z be Banach spaces, and let $U \subset X \times Y$ be a neighborhood of (λ_0, u_0). Let $F : U \to Z$ be a continuously differentiable mapping. Suppose that $F(\lambda_0, u_0) = 0$ and $F_u(\lambda_0, u_0)$ is an isomorphism, *i.e.* $F_u(\lambda_0, u_0)$ is one-to-one and onto, and $F_u^{-1}(\lambda_0, u_0) : Z \to Y$ is a linear bounded operator. Then there exists a neighborhood A of λ_0 in X, and a neighborhood B of u_0 in Y, such that for any $\lambda \in A$, there exists a unique $u(\lambda) \in B$ satisfying $F(\lambda, u(\lambda)) = 0$. Moreover $u(\cdot) : A \to B$ is continuously differentiable, and $u'(\lambda_0) : X \to Y$ is defined as $u'(\lambda_0)[\psi] = -[F_u(\lambda_0, u_0)]^{-1} \circ F_\lambda(\lambda_0, u_0)[\psi]$. If F is C^k, then $u : A \to B$ is also C^k.

History:
Implicit function:
[Descartes, 1637] [Newton, 1669] [Leibnitz, 1676] [J. Bernoulli, 1695]
Implicit function theorem in $\mathbb{R}^n$: [Dini, 1878]

Equivalent forms:
Inverse function theorem, Submersion theorem,
Constant rank theorem, Preimage theorem

Inverse function theorem

Theorem 3. Let X, Y be Banach spaces, and let $U \subset X$ be a neighborhood of u_0. Let $F : U \to Y$ be a continuously differentiable mapping. Suppose that $F(u_0) = v_0$ and the Frechét derivative $F'(u_0)$ is an isomorphism, *i.e.* $F'(u_0)$ is one-to-one and onto, and $[F'(u_0)]^{-1} : Y \to X$ is a linear bounded operator. Then there exists a neighborhood A of u_0 in U and a neighborhood B of v_0 in Y, such that $F : A \to B$ is an isomorphism, and for every $v \in B$, there exists a unique $u \in A$ such that $F(u) = v$. Hence $F^{-1} : B \to A$ is well-defined, and $(F^{-1})'(F(u)) = [F'(u)]^{-1}$ for $u \in A$. Moreover F^{-1} is as smooth as F.

Theorem 3a. (Finite dimensional) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^k smooth mapping such that $f(u_0) = 0$. Suppose that the matrix $f'(u_0) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exists a neighborhood A of u_0 and a neighborhood B of $f(u_0)$, such that for every $v \in B$, there exists a unique $u = u \in A$ such that $f(u) = v$. The inverse function $f^{-1} : B \to A$ is also C^k.

Proof. Define $G : U \times Y \to Y$ by $G(u, v) = F(u) - v$. Then $G(u_0, v_0) = 0$ and $G_u(u_0, v_0) = F'(u_0)$ is invertible. Then $G(u, v) = 0$ is solvable for $u = u(v)$ so that $G(u(v), v) = 0$.

Preimage Theorem

Theorem 4a. (finite dimensional) Suppose that $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is a C^k smooth mapping such that $f(u_0) = 0$. Suppose that the matrix $f'(u_0) : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is rank n (full rank). Then there exists a neighborhood A of u_0, such that the set $S = \{u \in A : f(u) = 0\}$ is a m-dimensional submanifold of A. (a special case is when $m = 1$, then S is a curve)

Let X, Y be Banach spaces.
1. A bounded linear mapping L from X to Y is Fredholm if the dimension of its kernel $\mathcal{N}(L)$ and the co-dimension of its range $\mathcal{R}(L)$ are both finite, and $\mathcal{R}(L)$ is closed. The Fredholm index of L is defined to be $ind(L) = dim\mathcal{N}(L) - codim\mathcal{R}(L)$.
2. A nonlinear mapping $F : X \to Y$ is Fredholm if its derivative $F' : X \to Y$ is Fredholm.
3. For any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$, $ind(L) = dim\mathcal{N}(L) - codim\mathcal{R}(L) = n - m$. (rank-nullity theorem in linear algebra)

Theorem 4. [Smale, 1964, AMJ] Let X, Y be Banach spaces, and let $U \subset X$ be a neighborhood of u_0. Let $F : U \to Y$ be a continuously differentiable mapping. Suppose that $F(u_0) = v_0$ and the Frechét derivative $F'(u_0)$ is Fredholm and surjective, then there exists a neighborhood A of u_0 in U, such that the set $S = \{u \in A : F(u) = 0\}$ is a manifold of dimension $ind(F'(u_0)) = dim\mathcal{N}(L)$.

Immersion and submersion

Let X, Y be Banach spaces, and let $U \subset X$ be a subset. Suppose that $f : U \to Y$ is a continuously differentiable mapping, and its derivative at $u_0 \in X$ is a Fredholm operator $f'(u_0) : X \to Y$.

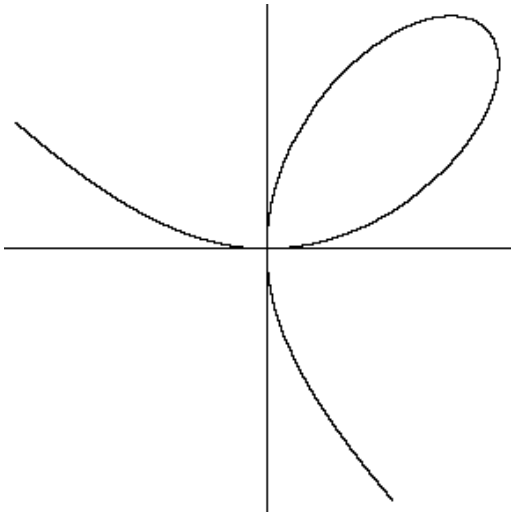
1. f is an submersion at u_0 if $f'(u_0)$ is surjective, *i.e.* $\mathcal{R}(f'(u_0)) = Y$. f is a submersion on U if it is a submersion at each $u_0 \in U$.
2. f is an immersion at u_0 if $f'(u_0)$ is injective, *i.e.* $\mathcal{N}(f'(u_0)) = \{0\}$. f is an immersion on U if it is an immersion at each $u_0 \in U$.

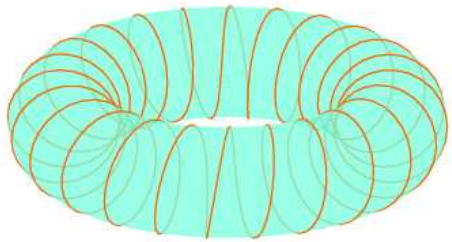
Submersion manifold: $f^{-1}(f(u_0))$ is locally a manifold of dimension $\dim \mathcal{N}(f'(u_0))$ containing u_0.

Immersion manifold: the image $f(U)$ is locally a Banach manifold of codimension $\operatorname{codim}(\mathcal{R}(f'(u_0)))$.

In general $f(U)$ may not be a true manifold. If $f : U \to Y$ is an immersion and injective, and U is compact, then $f(U)$ is a manifold. This is an embedding.

Examples





1. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^3 + y^3 - 3xy$. Then $f(x, y) = 0$ is the Folium of Descrates. It is locally a curve (except $(x, y) = (0, 0)$ since ∇f is surjective except at $(0, 0)$.
2. $f : \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (\cos t, \sin t)$. It is an immersion since $f'(t)$ is injective, and $f(\mathbb{R})$ is compact. In this case $f(\mathbb{R})$ is a manifold although f is not injective.
3. (irrational winding of a torus) The curve $y = \sqrt{2}x$ on $\mathbb{R}^2$ induces a curve on a torus $T = \mathbb{R}^2/\mathbb{Z}^2$. This curve is not closed on torus, and it is an immersion, but it is not an embedded manifold. Indeed it is dense on the torus. This is another example of the image of immersion may not be a manifold. This is does not have self-intersecting points as the first one.

Saddle-Node Bifurcation

Theorem 5 [Crandall-Rabinowitz, ARMA, 1973]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping. Assume that $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

(F2) $F_\lambda(\lambda_0, u_0) \notin \mathcal{R}(F_u(\lambda_0, u_0))$.

If Z is a complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X, then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function, $\lambda(0) = \lambda'(0) = 0$, and $z(0) = z'(0) = 0$.

Saddle-Node Bifurcation

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Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping. Assume that $F(\lambda_0, u_0) = 0$, F satisfies

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If Z is a complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X, then the solutions of $F(\lambda, u) = 0$ near (λ_0, u_0) form a curve
$\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbf{R} \times Z$ is a continuously differentiable function, $\lambda(0) = \lambda'(0) = 0$, and $z(0) = z'(0) = 0$.

Proof 1: Define $G(s, \lambda, z) = F(\lambda, u_0 + sw_0 + z)$ and prove that $G_{(\lambda,z)}(0, \lambda_0, 0) : \mathbb{R} \times Z \to Y$ is an isomorphism. Then apply the implicit function theorem.

Proof 2: Apply preimage theorem to $F(\lambda, u)$. Then $F'(\lambda_0, u_0)[\tau, w] = \tau F_\lambda(\lambda_0, u_0) + F_u(\lambda_0, u_0)[w]$ which is surjective. Also $F'(\lambda_0, u_0)[\tau, w] = 0$ has a 1-dimensional kernel $\{(0, w_0)\}$.

We will later consider the case

(F2') $F_\lambda(\lambda_0, u_0) \in \mathcal{R}(F_u(\lambda_0, u_0))$.

Turning direction

Condition **(F1)**: $\dim \mathcal{N}(F_u(\lambda_0, u_0)) = \operatorname{codim} \mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0 (\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $\mathcal{R}(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$
(Y^* is the dual space of Y, and $\langle l, v \rangle$ is the dual pair between Y^* and Y)

Turning direction

Condition **(F1)**: $\dim \mathcal{N}(F_u(\lambda_0, u_0)) = \operatorname{codim} \mathcal{R}(F_u(\lambda_0, u_0)) = 1$

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(Y^* is the dual space of Y, and $\langle l, v \rangle$ is the dual pair between Y^* and Y)

The solution curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbb{R} \times Z$ is C^1, $\lambda(0) = u_0$, $\lambda'(0) = 0$, $z(0) = z'(0) = 0$.

Turning direction

Condition **(F1)**: $\dim \mathcal{N}(F_u(\lambda_0, u_0)) = \operatorname{codim} \mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0 (\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $\mathcal{R}(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$
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The solution curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + z(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbb{R} \times Z$ is C^1, $\lambda(0) = u_0$, $\lambda'(0) = 0$, $z(0) = z'(0) = 0$.

If F is C^2 in u near (λ_0, u_0), then $\lambda(s)$ is C^2 at $\lambda = \lambda_0$, and

$$\lambda''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{\langle l, F_\lambda(\lambda_0, u_0) \rangle}.$$

$F(\lambda, u) = \Delta u + \lambda f(u)$, $F_u(\lambda, u)[w] = \Delta w + \lambda f'(u)w$,
$F_\lambda(\lambda, u) = f(u)$, $F_{uu}(\lambda, u)[\phi, \psi] = \lambda f'(u)\phi\psi$.
If $g \in \mathcal{R}(F_u(\lambda_0, u_0))$, $\Delta w + \lambda f'(u)w = g$ and $\int_\Omega g(x)w_0(x) = 0$
So $l : Y \to \mathbb{R}$ is defined by $l(g) = \int_\Omega g(x)w_0(x)0$.

Transcritical-Pitchfork Bifurcation

Theorem 6. [Crandall-Rabinowitz, 1971, JFA]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F : U \to Y$ be a continuously differentiable mapping such that $F_{\lambda u}$ exists and continuous in U. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

(F3) $F_{\lambda u}(\lambda_0, u_0)[w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$, where $w_0 \in \mathcal{N}(F_u(\lambda_0, u_0))$,

Let Z be any complement of $\mathcal{N}(F_u(\lambda_0, u_0)) = span\{w_0\}$ in X. Then the solution set of $F(\lambda, u) = 0$ near (λ_0, u_0) consists precisely of the curves $u = u_0$ and $\{(\lambda(s), u(s)) : s \in I = (-\epsilon, \epsilon)\}$, where $\lambda : I \to \mathbb{R}$, $z : I \to Z$ are continuous functions such that $u(s) = u_0 + sw_0 + sz(s)$, $\lambda(0) = \lambda_0$, $z(0) = 0$.

known as bifurcation from simple eigenvalue
[Chow and Hale, 1982, book] [Deimling, 1985, book]

Applications: (cited $>$ 600 times in MathSciNet)
Existence of steady periodic water waves
[Constantin-Strauss, 2004, CPAM], [Strauss, 2010, BAMS]
Free boundary problem in tumor models [Friedman-Hu, 2007, SIAM-MA]
Nonconstant stationary patterns in spatial ecological models
[Du-Hsu, 2010, SIAM-MA], [Shi-Shivaji, 2006, JMB]

Turning direction

Condition **(F1)**: $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0 (\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$

Turning direction

Condition **(F1)**: $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0 (\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v \rangle = 0\}$

The nontrivial solution curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + sz(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbb{R} \times Z$ is C^0, $\lambda(0) = u_0$, $z(0) = 0$, and if F is C^2 in u

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0] \rangle}.$$

So if

(F4) $F_{uu}(\lambda_0, u_0)[w_0, w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$,

is satisfied, then a transcritical bifurcation occurs.

Turning direction

Condition **(F1)**: $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$

$X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$
$Y = \mathcal{R}(F_u(\lambda_0, u_0)) \oplus Y_1$
$w_0 (\neq 0) \in \mathcal{N}(F_u(\lambda_0, u_0))$
$F_u(\lambda_0, u_0)|_Z : Z \to \mathcal{R}(F_u(\lambda_0, u_0))$ is an isomorphism
there exists $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$

The nontrivial solution curve $\{(\lambda(s), u(s)) = (\lambda(s), u_0 + sw_0 + sz(s)) : |s| < \delta\}$, where $s \mapsto (\lambda(s), z(s)) \in \mathbb{R} \times Z$ is C^0, $\lambda(0) = u_0$, $z(0) = 0$, and if F is C^2 in u

$$\lambda'(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle}{2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle}.$$

So if
(F4) $F_{uu}(\lambda_0, u_0)[w_0, w_0] \notin \mathcal{R}(F_u(\lambda_0, u_0))$,
is satisfied, then a transcritical bifurcation occurs.

If **(F4)** is not satisfied, and F is C^3, then (θ satisfies $F_{uu}(\lambda_0, u_0)[w_0, w_0] + F_u(\lambda_0, u_0)[\theta] = 0$)

$$\lambda''(0) = -\frac{\langle l, F_{uuu}(\lambda_0, u_0)[w_0, w_0, w_0]\rangle + 3\langle l, F_{uu}(\lambda_0, u_0)[w_0, \theta]\rangle}{3\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle},$$

and if $\lambda''(0) \neq 0$, then a pitchfork bifurcation occurs.

Crossing Curve Bifurcation

Theorem 7. [Liu, Shi and Wang, 2007, JFA]
Let $F : \mathbb{R} \times X \to Y$ be a C^2 mapping. Suppose that $F(\lambda_0, u_0) = 0$, F satisfies

(F1) $dim\mathcal{N}(F_u(\lambda_0, u_0)) = codim\mathcal{R}(F_u(\lambda_0, u_0)) = 1$, and

(F2') $F_\lambda(\lambda_0, u_0) \in \mathcal{R}(F_u(\lambda_0, u_0))$.

Let $X = \mathcal{N}(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, let $v_1 \in Z$ be the unique solution of $F_\lambda + F_u[v] = 0$, and let $l \in Y^*$ such that $\mathcal{R}(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$.
We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H_0 = H_0(\lambda_0, u_0) \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1]^2\rangle & \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle \\ \langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle & \langle l, F_{uu}[w_0]^2\rangle \end{pmatrix}$$

is non-degenerate, i.e., $\det(H_0) \neq 0$.

1. If H_0 is definite, *i.e.* $\det(H_0) > 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is $\{(\lambda_0, u_0)\}$.
2. If H_0 is indefinite, *i.e.* $\det(H_0) < 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of two intersecting C^1 curves, and the two curves are in form of $(\lambda_i(s), u_i(s)) = (\lambda_0 + \mu_i s + s\theta_i(s), u_0 + \eta_i s w_0 + s y_i(s))$, $i = 1, 2$, where $s \in (-\delta, \delta)$ for some $\delta > 0$, (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

$$\begin{aligned} &\langle l, F_{\lambda\lambda} + 2F_{\lambda u}[v_1] + F_{uu}[v_1]^2\rangle\mu^2 + 2\langle l, F_{\lambda u}[w_0] + F_{uu}[w_0, v_1]\rangle\eta\mu \\ &+ \langle l, F_{uu}[w_0]^2\rangle\eta^2 = 0, \end{aligned}$$

where $\theta_i(s), y_i(s)$ are some functions defined on $s \in (-\delta, \delta)$ which satisfy $\theta_i(0) = \theta_i'(0) = 0$, $y_i(s) \in Z$, and $y_i(0) = y_i'(0) = 0$, $i = 1, 2$.

Remarks

1. When $F_\lambda(\lambda_0, u_0) = 0$, we have $v_1 = 0$.
$$H_1 \equiv \begin{pmatrix} \langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle & \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle \\ \langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle & \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle \end{pmatrix}$$
and the equation of tangents of curves become
$$\langle l, F_{\lambda\lambda}(\lambda_0, u_0)\rangle\mu^2 + 2\langle l, F_{\lambda u}(\lambda_0, u_0)[w_0]\rangle\mu\eta + \langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0]\rangle\eta^2 = 0.$$
2. If $F(\lambda, u_0) \equiv 0$, then Theorem 6 (classical transcritical and pitchfork bifurcation theorem) follows from Theorem 7.
3. Theorem 7 is a natural complement Crandall-Rabinowitz saddle-node bifurcation theorem (Theorem 5), where $(F2)$ is imposed. Our result is based on condition the opposite $(F2')$ and a generic second order non-degeneracy condition $\det(H_0) \neq 0$.
4. An example for single point solution set is $F(\lambda, u) = \lambda^2 + u^2$. Then $(\lambda, u) = (0, 0)$ is the only solution (also a degenerate solution)

Secondary Bifurcation

Theorem 8. [Liu-Shi-Wang, 2007, JFA] [Crandall-Rabinowitz, 1971, JFA]
Let X and Y be Banach spaces, let U be an open subset of X and let $F : U \to Y$ be twice differentiable. Suppose

1. $F(u_0) = 0$,
2. $\dim \mathcal{N}(F'(u_0)) = 2$, $\operatorname{codim} \mathcal{R}(F'(u_0)) = 1$.

Then

1. if for any $\phi(\neq 0) \in \mathcal{N}(F'(u_0))$, $F''(u_0)[\phi, \phi] \notin \mathcal{R}(F'(w_0))$, then the set of solutions to $F(u) = 0$ near $u = u_0$ is the singleton $\{u_0\}$.
2. if there exists $\phi_1(\neq 0) \in \mathcal{N}(F'(u_0))$ such that $F''(u_0)[\phi_1]^2 \in \mathcal{R}(F'(u_0))$, and there exists $\phi_2 \in \mathcal{N}(F'(u_0))$ such that $F''(u_0)[\phi_1, \phi_2] \notin \mathcal{R}(F'(u_0))$, then u_0 is a bifurcation point of $F(u) = 0$ and in some neighborhood of u_0, the totality of solutions of $F(u) = 0$ form two differentiable curves intersecting only at u_0. Moreover the solution curves are in form of $u_0 + s\psi_i + s\theta_i(s)$, $s \in (-\delta, \delta)$, $\theta_i(0) = \theta_i'(0) = 0$, where ψ_i $(i = 1, 2)$ are the two linear independent solutions of the equation $\langle l, F''(u_0)[\psi, \psi] \rangle = 0$ and $l \in Y^*$ satisfies $\mathcal{N}(l) = \mathcal{R}(F'(u_0))$.

Summary of bifurcation results

Bifurcation problem: $F(\lambda, u) = 0$, assuming $F(\lambda_0, u_0) = 0$, in all case
$\dim \mathcal{N}(F_u(\lambda_0, u_0)) = \text{codim}(F_u(\lambda_0, u_0))$ (Fredholm index 0)

Name	$\dim \mathcal{N}$	solution set	Trans-cond	theorem
implicit function	0	1 monotone curve	none	Theorem 1
saddle-node	1	1 turning curve	$F_\lambda \notin \mathcal{R}$	Theorem 5
transcritical, pitchfork	1	2 crossing curves	$F_{\lambda u}[w_0] \notin \mathcal{R}$	Theorem 6
crossing-curve	1	2 crossing curves	$detH < 0$	Theorem 7
degenerate simple	1	3 crossing curves	$F_{\lambda u}[w_0] \in \mathcal{R}$, more	Theorem 9
double saddle-node	2	2 tangent curves	$F_\lambda \notin \mathcal{R}$	Theorem 10

Mapping (no parameter): $F(u) = 0$, assuming $F(u_0) = 0$

Name	$\dim \mathcal{N}$	$\text{codim}\, \mathcal{R}$	solution set	Theorem
inverse function	0	0	one point	Theorem 3
preimage	1	0	a curve	Theorem 4
preimage	k	0	a k-d manifold	Theorem 4
secondary	2	1	two crossing curves	Theorem 8

Degenerate bifurcation from simple eigenvalue

Theorem 9. [Liu-Shi-Wang, 2013, JFA]
Let U be a neighborhood of (λ_0, u_0) in $\mathbb{R} \times X$, and let $F \in C^3(U, Y)$. Assume that $F(\lambda, u_0) = 0$ for $(\lambda, u_0) \in U$. At (λ_0, u_0), F satisfies **(F1)**,

(F3′) $F_{\lambda u}(\lambda_0, u_0)[w_0] \in \mathcal{R}(F_u(\lambda_0, u_0))$; and

(F4′) $F_{uu}(\lambda_0, u_0)[w_0]^2 \in \mathcal{R}(F_u(\lambda_0, u_0))$.

Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, and let $l \in Y^*$ such that $R(F_u(\lambda_0, u_0)) = \{v \in Y : \langle l, v\rangle = 0\}$. Denote by $v_1 \in Z$ the unique solution of $F_{\lambda u}(\lambda_0, u_0)[w_0] + F_u(\lambda_0, u_0)[v] = 0$, and $v_2 \in Z$ the unique solution of $F_{uu}(\lambda_0, u_0)[w_0]^2 + F_u(\lambda_0, u_0)[v] = 0$. We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H = H(\lambda_0, u_0) = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$

is non-degenerate, *i.e.*, $\det(H) \neq 0$, where H_{ij} is given by

$$H_{11} = \langle l, F_{\lambda\lambda u}[w_0] + 2F_{\lambda u}[v_1]\rangle,\ H_{22} = \frac{1}{3}\langle l, F_{uuu}[w_0]^3 + 3F_{uu}[w_0, v_2]\rangle,$$

$$H_{12} = \frac{1}{2}\langle l, F_{\lambda uu}[w_0]^2 + F_{\lambda u}[v_2] + 2F_{uu}[w_0, v_1]\rangle$$

Degenerate bifurcation from simple eigenvalue

1. If H is definite, *i.e.* $\det(H) > 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the line $\{(\lambda, u_0)\}$.
2. If H is indefinite, *i.e.* $\det(H) < 0$, then the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$ is the union of C^1 curves intersecting at (λ_0, u_0), including the line of trivial solutions $\Gamma_0 = \{(\lambda, u_0)\}$ and two other curves $\Gamma_i = \{(\lambda_i(s), u_i(s)) : |s| < \delta\}$ $(i = 1, 2)$ for some $\delta > 0$, with

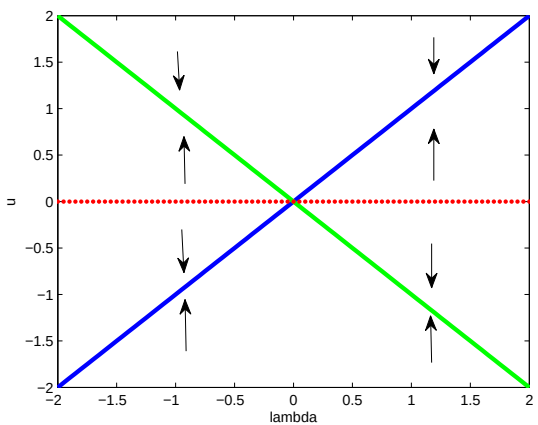
$$\lambda_i(s) = \lambda_0 + \mu_i s + s\theta_i(s), \quad u_i(s) = u_0 + \eta_i s w_0 + s v_i(s),$$

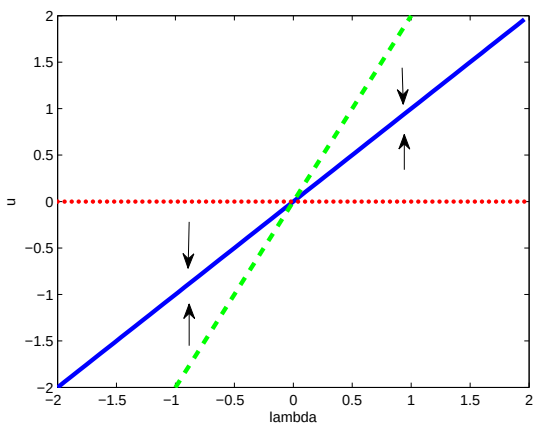
where (μ_1, η_1) and (μ_2, η_2) are non-zero linear independent solutions of the equation

$$H_{11}\mu^2 + 2H_{12}\mu\eta + H_{22}\eta^2 = 0,$$

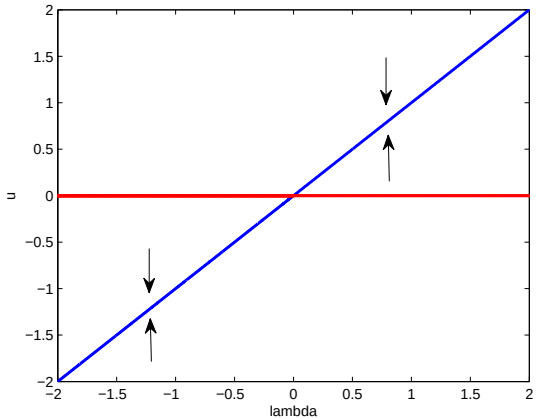
$\theta_i(0) = \theta_i'(0) = 0$, $v_i(s) \in Z$, and $v_i(0) = v_i'(0) = 0$, $i = 1, 2$.

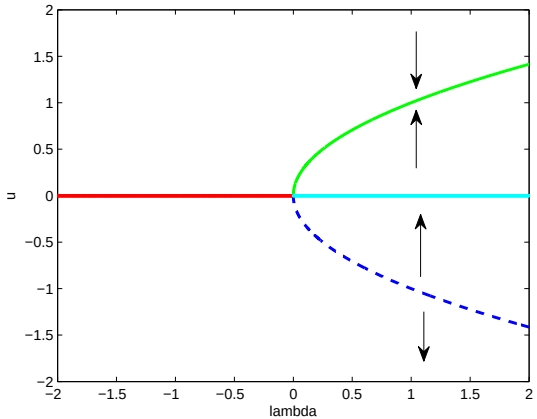
Example: (Left):$F(\lambda, u) = u(\lambda^2 - u^2)$; (Right):$F(\lambda, u) = u(u - \lambda)(u - 2\lambda)$.



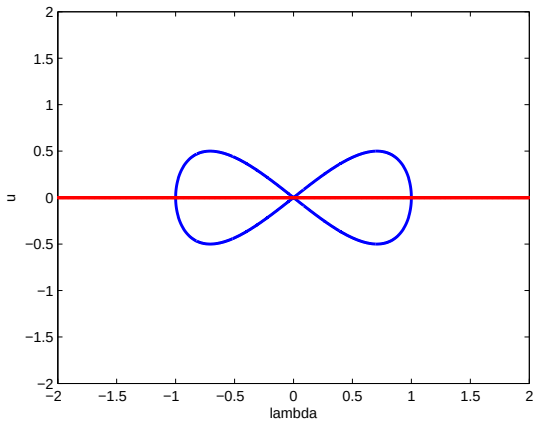


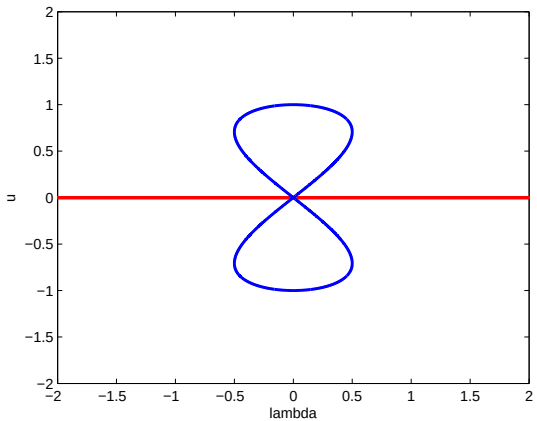
More examples





(Left):$F(\lambda, u) = \lambda u^2 - u^3$; (Right):$F(\lambda, u) = \lambda u^2 - u^4$.





(Left): $F(\lambda, u) = u(\lambda^2 - u^2 + u^4)$; (Right): $F(\lambda, u) = u(\lambda^4 - \lambda^2 + u^2)$.

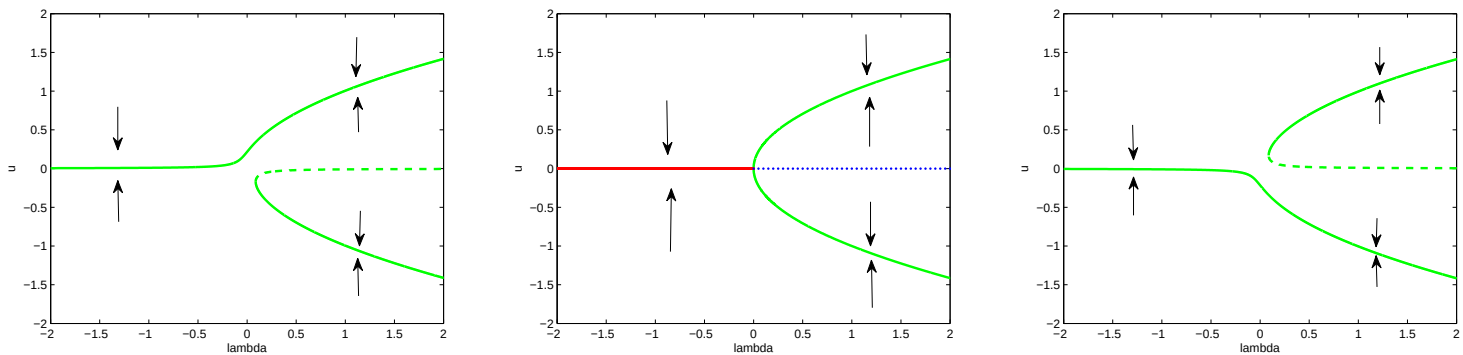
Application: perturbation of diffusive logistic equation

Assume that $\lambda > 0$, $\varepsilon \in \mathbb{R}$, Ω is a bounded region with smooth boundary in $\mathbb{R}^n$ for $n \geq 1$, $a(\cdot), b(\cdot), c(\cdot) \in C^\alpha(\overline{\Omega})$, and

$$\int_\Omega a(x)\phi_1^2(x)dx > 0, \quad \int_\Omega b(x)\phi_1^4(x)dx > 0, \quad \text{and} \quad \int_\Omega c(x)\phi_1^3(x)dx > 0.$$

[Liu-Shi-Wang, 2007, JFA]

$$\begin{cases} \Delta u + \lambda[a(x)u - b(x)u^3 - \varepsilon c(x)] = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$



(Left):$\varepsilon < 0$; (Middle): $\varepsilon = 0$; (Right):$\varepsilon > 0$.

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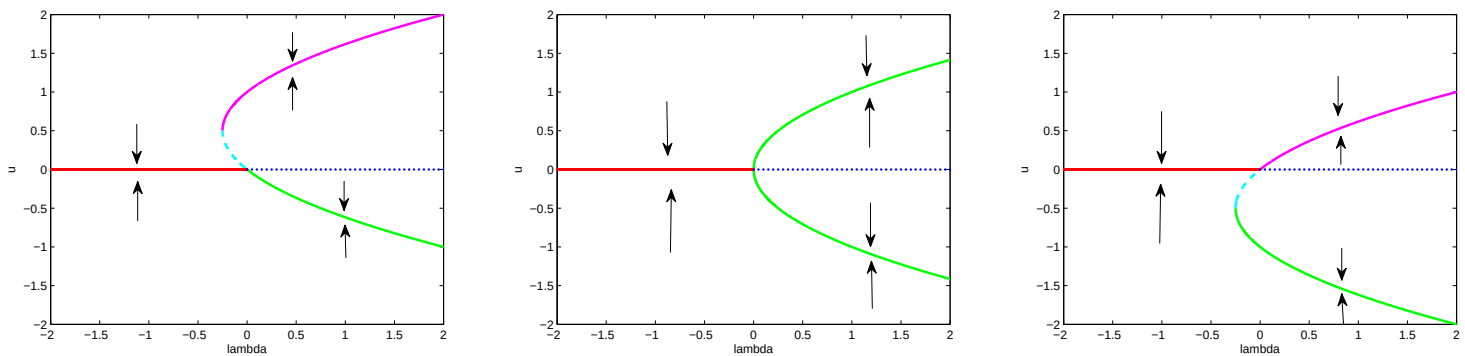

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What if Kernel is 2-dimensional?

Let $F : \mathbb{R} \times X \to Y$ be continuously differentiable. $F(\lambda_0, u_0) = 0$, F satisfies
(F1-2) $dim N(F_u(\lambda_0, u_0)) = codim R(F_u(\lambda_0, u_0)) = 2$, and
(F2) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.
and additional non-degeneracy condition on D^2F

Then a saddle-node bifurcation of two curves occurs
[Liu-Shi-Wang, 2013, CPAA]

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If F satisfies **(F1-2)**,
(F2') $F_\lambda(\lambda_0, u_0) \in R(F_u(\lambda_0, u_0))$,
more complicated, depending on the symmetry of the problem
(example: two-dimensional surface, four curves)

Double Saddle-Node Bifurcation

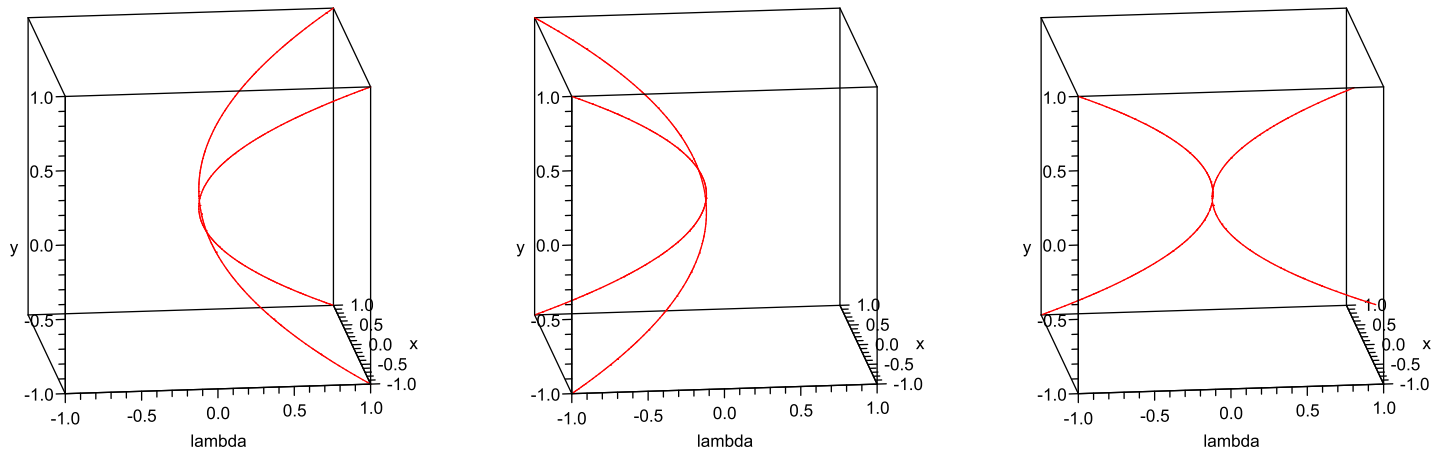
Theorem 10 [Liu-Shi-Wang, 2013, CPAA]
Let $F : \mathbb{R} \times X \to Y$ be a C^p mapping, and assume (F1-2) and (F2). Let $X = N(F_u(\lambda_0, u_0)) \oplus Z$ be a fixed splitting of X, and let $v_1, v_2 \in Y^*$ such that $R(F_u) = \{h \in Y : \langle v_1, h\rangle = 0 \text{ and } \langle v_2, h\rangle = 0\}$ so that $\langle v_1, F_\lambda\rangle \neq 0$ and $\langle v_2, F_\lambda\rangle = 0$. We assume that the matrix (all derivatives are evaluated at (λ_0, u_0))

$$H_2 = H_2(\lambda_0, u_0) \equiv \begin{pmatrix} \langle v_2, F_{uu}[w_1, w_1]\rangle & \langle v_2, F_{uu}[w_1, w_2]\rangle \\ \langle v_2, F_{uu}[w_1, w_2]\rangle & \langle v_2, F_{uu}[w_2, w_2]\rangle \end{pmatrix}$$

is non-degenerate, i.e., $\det(H_2) \neq 0$. Let S be the solution set of $F(\lambda, u) = 0$ near $(\lambda, u) = (\lambda_0, u_0)$.

1. If H_2 is definite, *i.e.* $\det(H_2) > 0$, then S is $\{(\lambda_0, u_0)\}$.
2. If H_2 is indefinite, *i.e.* $\det(H_2) < 0$, then S is the union of two C^{p-1} tangent curves.

An Example



Three types of double saddle-node bifurcations: (left) $\lambda - x^2 = 0$ and $\lambda - y^2 = 0$, supercritical; (middle) $\lambda - x^2 + 2y^2 = 0$ and $\lambda - y^2 + 2x^2 = 0$, subcritical; (right) $\lambda - x^2 - 2xy = 0$ and $\lambda - y^2 - 2xy = 0$, transcritical.

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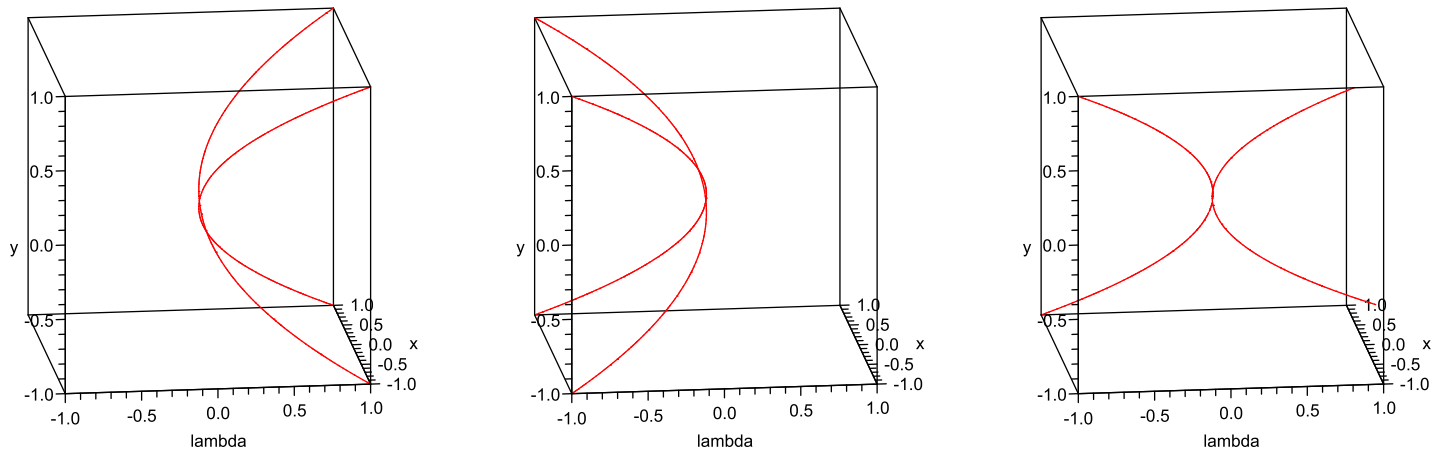

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Question: Is there a realistic example of double saddle-node bifurcation?

An example of double saddle-node bifurcation

[Zhao-Zhao-Shi, 2015, CVPDE]

$$\begin{cases} \Delta v - v + vw = 0, & x \in \Omega, \\ \Delta w - \alpha w + \frac{1}{2} v^2 = 0, & x \in \Omega, \\ v = w = 0, & x \in \partial\Omega. \end{cases} \tag{1}$$

Let (λ_1, φ_1) be the principal eigen-pair of $-\Delta$ on $H_0^1(\Omega)$. Then it is well-known that $\lambda_1 > 0$ is simple and φ_1 can be chosen as positive. We observe that when $\alpha = -\lambda_1$, the system (1) has a branch of semi-trivial solutions $\Sigma_* = \{(\alpha, v, w) = (-\lambda_1, 0, \mu\varphi_1) : \mu \in \mathbb{R}\}$ which intersects with the branch of the trivial solutions $\Sigma_0 = \{(\alpha, 0, 0) : \alpha \in \mathbb{R}\}$ at $(-\lambda_1, 0, 0)$.

We consider an eigenvalue problem:

$$\begin{cases} \Delta\phi - \phi + \mu\varphi_1\phi = 0, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

Then (2) has a sequence of eigenvalues $0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_k \leq \cdots \to \infty$, the eigenfunction ϕ_1 corresponding to μ_1 is simple, and ϕ_1 can be chosen as positive. The principal eigenvalue μ_1 can be expressed by the following Rayleigh quotient:

$$\mu_1 = \inf_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (|\nabla\phi|^2 + \phi^2)\, dx}{\int_\Omega \varphi_1 \phi^2\, dx}.$$

Bifurcation result

Let $X = C_0^{2,\theta}(\overline{\Omega})$ for $\theta \in (0,1)$. Define the set of non-trivial solutions of (1) to be

$$\Sigma = \{(\alpha, v, w) \in \mathbb{R} \times X^2 : (\alpha, v, w) \text{ is a solution of (1) such that } v \not\equiv 0 \text{ and } w \not\equiv 0\}. \tag{3}$$

Theorem. Suppose that $N \geq 1$ and $\Omega \subset \mathbb{N}$ is a bounded domain with smooth boundary $\partial\Omega$.

1. For $-\infty < \alpha < -\lambda_1$, the only non-negative solution of system (1) is $(v, w) = (0, 0)$.
2. For $\alpha = -\lambda_1$, if (v, w) is a solution of (1), then $v = 0$ and $w = \mu\varphi_1$ for $\mu \in \mathbb{R}$, that is $(-\lambda_1, v, w) \in \Sigma_*$.
3. Assume that all eigenvalues μ_n of (2) are simple ones. Then for any $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that for $-\lambda_1 < \alpha < -\lambda_1 + \delta_n$, system (1) possesses a pair of non-trivial solutions in form of $(\pm v_n(\alpha, \cdot), w_n(\alpha, \cdot))$ such that

$$v_n(\alpha, \cdot) = k_n(\alpha + \lambda_1)^2\phi_n + o(|\alpha + \lambda_1|), \; w_n(\alpha, \cdot) = \mu_n\varphi_1 + o(|\alpha + \lambda_1|), \tag{4}$$

for some constant $k_n > 0$. Moreover for each $n \in \mathbb{N}$, there exists a connected component Σ_n of $\overline{\Sigma}$ such that $(-\lambda_1, 0, \mu_n\varphi_1) \in \Sigma_n$, and either Σ_n is unbounded in $(-\lambda_1, \infty) \times X^2$, or there exists $m \in \mathbb{N}$ and $m \neq n$ such that $(-\lambda_1, 0, \mu_m\varphi_1) \in \Sigma_n$, that is, $\Sigma_n = \Sigma_m$.

Bifurcation diagram

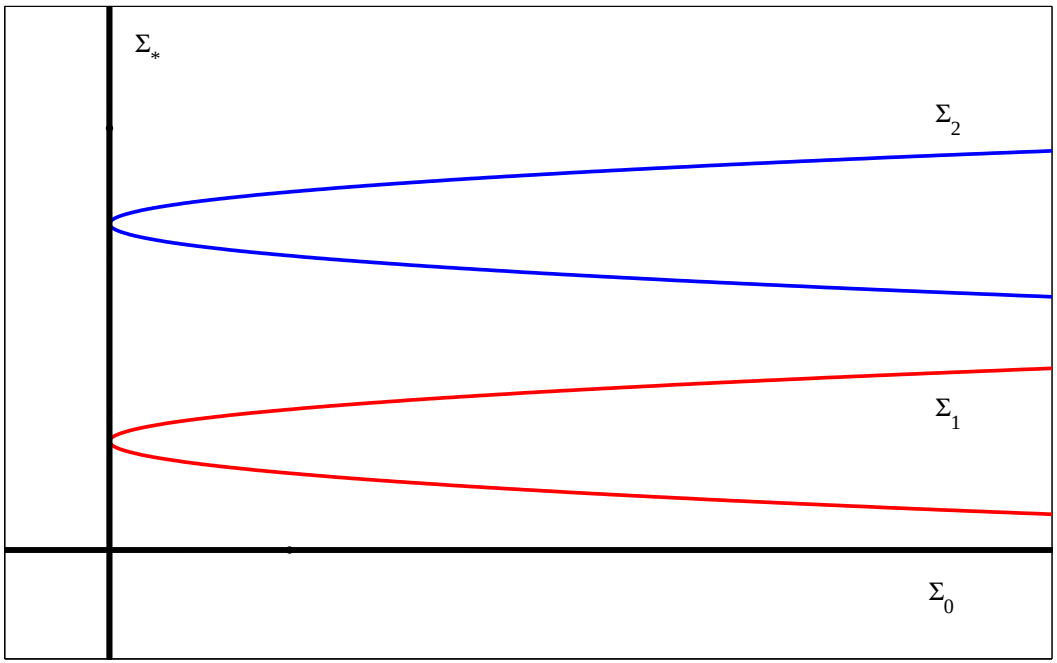

Figure: Illustration of global bifurcation diagram of (1). Here the horizontal direction is μ, and the vertical direction is (v, w).

Lyapunov-Schmidt reduction

Suppose that $F : \mathbb{R} \times X \to Y$ is a C^p map such that $F(\lambda_0, u_0) = y_0$, and F satisfies $(F1)$ at (λ_0, u_0). Then $F(\lambda, u) = y_0$ for (λ, u) near (λ_0, u_0) can be reduced to $\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$, where $t \in (-\delta, \delta)$, $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$ where δ is a small constant,, $l \in Y^*$ such that $\langle l, u\rangle = 0$ if and only if $u \in \mathcal{R}(F_u(\lambda_0, u_0))$, and g is a C^p function into Z such that $g(\lambda_0, 0) = 0$ and Z is a complement of $N(F_u(\lambda_0, u_0))$ in X.

Let the projection from Y into $R(F_u(\lambda_0, u_0))$ be Q. Then $F(\lambda, u) = 0$ is equivalent to

$$Q \circ F(\lambda, u) = 0, \quad \text{and } (I - Q) \circ F(\lambda, u) = 0.$$

We rewrite the first equation in form

$$G(\lambda, t, g) \equiv Q \circ F(\lambda, u_0 + tw_0 + g) = 0$$

where $t \in \mathbb{R}$ and $g \in Z$. Calculation shows that $G_g(\lambda_0, 0, 0) = Q \circ F_u(\lambda_0, u_0)$ is an isomorphism from Z to $R(F_u(\lambda_0, u_0))$. Then $g = g(\lambda, t)$ is uniquely solvable from the implicit function theorem for (λ, t) near $(\lambda_0, 0)$, and g is C^2. Hence $u = u_0 + tw_0 + g(\lambda, t)$ is a solution to $F(\lambda, u) = 0$ if and only if $(I - Q) \circ F(\lambda, u_0 + tw_0 + g(\lambda, t)) = 0$. Since $R(F_u(\lambda_0, u_0))$ is co-dimensional one, hence it becomes the scalar equation $\langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$.

Proof of Main Results

1. Lyapunov-Schmidt reduction: reduce $F(\lambda, u) = 0$ to

$G(\lambda, t) \equiv \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$

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2. A finite dimensional result (improving result based on Morse lemma)

Morse lemma: [Nirenberg, 1974, book] Suppose that $f : \mathbb{R}^k \to \mathbb{R}$ is a C^p function, $k \geq 2$. If $f(0) = 0$, $f_x(0) = 0$, and the Hessian $f_{xx}(0)$ is a non-degenerate $k \times k$ matrix. Then there exists a local C^{p-2} coordinate change $y(x)$ defined in a neighborhood of the origin with $y(0) = 0$, $y_x(0) = I$ such that

$$f(x) = \frac{1}{2} y(x)^T f_{xx}(0) y(x),$$

where $y(x)^T$ is the transpose of $y(x)$, and $y(x)$ is assumed to be column vector in $\mathbb{R}^k$. In particular if $k = 2$ and $f_{xx}(0)$ is indefinite, then the set of solutions of $f(x) = 0$ near the origin consists of two C^{p-2} curves intersecting only at the origin.

A Finite Dimensional Theorem

[Liu-Wang-Shi, 2007, JFA]
Suppose that $(x_0, y_0) \in \mathbb{R}^2$ and U is a neighborhood of (x_0, y_0). Assume that $f : U \to \mathbb{R}$ is a C^p function for $p \geq 2$, $f(x_0, y_0) = 0$, $\nabla f(x_0, y_0) = 0$, and the Hessian $H = H(x_0, y_0)$ is non-degenerate. Then

1. If H is definite, then (x_0, y_0) is the unique zero point of $f(x, y) = 0$ near (x_0, y_0);
2. If H is indefinite, then there exist two C^{p-1} curves $(x_i(t), y_i(t))$, $i = 1, 2$, $t \in (-\delta, \delta)$, such that the solution set of $f(x, y) = 0$ consists of exactly the two curves near (x_0, y_0), $(x_i(0), y_i(0)) = (x_0, y_0)$. Moreover t can be rescaled and indices can be rearranged so that $(x_1'(0), y_1'(0))$ and $(x_2'(0), y_2'(0))$ are the two linear independent solutions of

$$f_{xx}(x_0, y_0)\eta^2 + 2f_{xy}(x_0, y_0)\eta\tau + f_{yy}(x_0, y_0)\tau^2 = 0.$$

Proof of the "Calculus Problem"

Consider

$$x' = \frac{\partial f(x,y)}{\partial y}, \quad y' = -\frac{\partial f(x,y)}{\partial x}, \quad (x(0), y(0)) \in U.$$

Then it is a Hamiltonian system with potential function $f(x, y)$, (x_0, y_0) is the only equilibrium point in U, and (x_0, y_0) is a saddle point. From the invariant manifold theory of differential equations, the set $\{f(x, y) = 0\}$ near (x_0, y_0) is consisted of the 1-dimensional stable and unstable manifolds at (x_0, y_0), which are C^{p-1} since f is C^p.

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Question: is there a proof without using invariant manifold theory but only elementary calculus?

another more general theorem:

splitting lemma [Kuiper, 1972] [Chang, 1993, book] [Li-Li-Liu, 2005, JFA]

Complete the Proof

Proof of Theorem 7 (crossing-curves)

Apply the 2-D lemma to the bifurcation equation
$G(\lambda, t) \equiv \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$.

Proof of Theorem 9 (degenerate simple eigenvalue)

$G(\lambda, t) \equiv \langle l, F(\lambda, u_0 + tw_0 + g(\lambda, t))\rangle = 0$ has a line of trivial solutions: $t = 0$ for all λ. So we apply the 2-D lemma to the function $h(\lambda, t) = \begin{cases} \frac{1}{t} G(\lambda, t), & \text{if } t \neq 0, \\ G_t(\lambda, 0), & \text{if } t = 0. \end{cases}$

Proof of theorem 10 (double saddle-node)

Since the kernel is 2-D, then the Lyapunov-Schmidt reduction reduces it to

$G_i(\lambda, s_1, s_2) = \langle v_i, F(\lambda, u_0 + s_1 w_1 + s_2 w_2 + g(\lambda, s_1, s_2))\rangle, \ i = 1, 2.$

But λ can be solved by (s_1, s_2) by implicit function theorem (since $F_\lambda \notin \mathcal{R}(F_u(\lambda_0, u_0))$). Then we can apply the 2-D lemma to $f_2(s_1, s_2) \equiv G_1(\lambda(s_1, s_2), s_1, s_2)$.

Stability

In the bifurcation from simple eigenvalue described in Theorem 6, an exchange of stability occurs at the bifurcation point between the known trivial solutions and the bifurcating nontrivial solutions.

Let $T, K \in B(X, Y)$ (the set of bounded linear maps from X into Y). We say that $\mu \in \mathbb{R}$ is a K-simple eigenvalue of T, if

$$\dim N(T - \mu K) = \operatorname{codim} R(T - \mu K) = 1, \quad N(T - \mu K) = \operatorname{span}\{w_0\},$$

and

$$K[w_0] \notin R(T - \mu K).$$

Lemma[Crandall and Rabinowitz, 1973]: Suppose that $T_0, K \in B(X, Y)$ and μ_0 is a K-simple eigenvalue of T_0. Then there exists $\delta > 0$ such that if $T \in B(X, Y)$ and $\|T - T_0\| < \delta$, then there exists a unique $\mu(T) \in \mathbb{R}$ satisfying $\|\mu(T) - \mu_0\| < \delta$ such that $N(T - \mu(T)K) \neq \emptyset$ and $\mu(T)$ is a K-simple eigenvalue of T. Moreover if $N(T_0 - \mu_0 K) = \operatorname{span}\{w_0\}$ and Z is a complement of $\operatorname{span}\{w_0\}$ in X, then there exists a unique $w(T) \in X$ such that $N(T - \mu(T)K) = \operatorname{span}\{w(T)\}$, $w(T) - w_0 \in Z$ and the map $T \mapsto (\mu(T), w(T))$ is analytic.

Exchange of stability

Theorem [Crandall and Rabinowitz, 1973] Let X, Y, U, F, Z, λ_0, and w_0 be the same as in Theorem 2, and let all assumptions in Theorem 2 on F be satisfied. Let $(\lambda(t), u(t))$ be the curve of nontrivial in Theorem 2, there exist C^2 functions $m(\lambda) : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to \mathbb{R}$, $z : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to X$, $\mu : (-\delta, \delta) \to \mathbb{R}$, and $w : (-\delta, \delta) \to X$, such that

$$F_u(\lambda, u_0)z(\lambda) = m(\lambda)K(z(\lambda)), \quad \text{for} \quad \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon), \tag{5}$$

$$F_u(\lambda(t), u(t))\phi(t) = \mu(t)K(\phi(t)), \quad \text{for} \quad t \in (-\delta, \delta). \tag{6}$$

where $m(\lambda_0) = \mu(0) = 0$, $z(\lambda_0) = w(0) = w_0$. Moreover, near $t = 0$ the functions $\mu(t)$ and $-t\lambda'(t)m'(\lambda_0)$ have the same zeroes and, whenever $\mu(t) \neq 0$ the same sign and satisfy

$$\lim_{t \to 0} \frac{-t\lambda'(t)m'(\lambda_0)}{\mu(t)} = 1. \tag{7}$$

Direction of bifurcations

A saddle-node bifurcation is supercritical (forward) if $\lambda''(0) > 0$, and it is subcritical (backward) if $\lambda''(0) < 0$.

A pitchfork bifurcation is supercritical (forward) if $\lambda''(0) > 0$, and it is subcritical (backward) if $\lambda''(0) < 0$.

In a transcritical bifurcation, it is often that only the positive portion (the part with $s \in (0, \delta)$) of curve is concerned). If the solutions with $s \in (0, \delta)$ are stable, then it is forward transcritical bifurcation; and if the solutions with $s \in (0, \delta)$ are unstable, then it is backward transcritical bifurcation.

An ODE system of epidemic model

Adapted from [Wang Wending, 2006, Math. Bios.]

$$S' = A - dS - \lambda SI, \quad I' = \lambda SI - \gamma I - \frac{kI}{1+I}$$

parameters A: recruitment rate, d, γ: mortality rates, λ: infection rate, $T(I)$: treatment

Trivial equilibrium: $(S, I) = (A/d, 0)$. Define $F(\lambda, S, I) = \begin{pmatrix} A - dS - \lambda SI \\ \lambda SI - \gamma I - \frac{kI}{1+I} \end{pmatrix}$.

$$F_{(S,I)} = \begin{pmatrix} -d - \lambda I & -\lambda S \\ \lambda I & \lambda S - \gamma - \frac{ka}{(a+I)^2} \end{pmatrix},$$

$$F_{(S,I)}(\lambda, A/d, 0) = \begin{pmatrix} -d & -\lambda A/d \\ 0 & \lambda A/d - \gamma - k/a \end{pmatrix}$$

So bifurcation point $\lambda = \lambda_* = \dfrac{\gamma + k/a}{A/d} = \dfrac{d(a\gamma + k)}{Aa}$.

Eigenvector $w_0 = [\phi, \psi] = (-\lambda_* A, d^2)$
$\mathcal{R}F_{(S,I)}(\lambda, A/d, 0) = \{(a, b) \in \mathbb{R}^2 : b = 0\} = \{(a, b) \in \mathbb{R}^2 : (a, b) \cdot (0, 1) = 0\}$.

An ODE system of epidemic model

$$F_{(S,I)(S,I)}[(\phi,\psi]^2 = \begin{pmatrix} -2\lambda\phi\psi \\ 2\lambda\phi\psi + 2k\psi^2/a \end{pmatrix}$$

$$F_{\lambda(S,I)}[(\phi,\psi] = \begin{pmatrix} -A\psi_1/d \\ A\psi_1/d \end{pmatrix}$$

$$\lambda'(0) = -\frac{\langle I, F_{(S,I)(S,I)}(\lambda_0, A/d, 0)[w_0, w_0]\rangle}{2\langle I, F_{\lambda(S,I)}(\lambda_0, A/d, 0)[w_0]\rangle} = \frac{d^3[(a\gamma + k)^2 - kAa]}{a^2A^2}.$$

So a backward bifurcation ($\lambda'(0) < 0$) occurs if A is large.

Final Remark

If we do not succeed in solving a mathematical problem then, very often, the reason is that we did not yet discover the more general point of view from which the given problem appears to be a link in a chain of related problems. Having found this point of view, not only the given problem becomes more accessible to our research, but we also gain a method which is applicable to related problems $\cdots$

Final Remark

If we do not succeed in solving a mathematical problem then, very often, the reason is that we did not yet discover the more general point of view from which the given problem appears to be a link in a chain of related problems. Having found this point of view, not only the given problem becomes more accessible to our research, but we also gain a method which is applicable to related problems $\cdots$
In dealing with mathematical problems, specializing plays—as I believe—an even more important role than generalizing. Perhaps in most cases in which we fail to find an answer, the reason for this failure is that we did not solve, at least not completely, problems simpler and easier than the given one. Everything amounts to finding these easier problems and to solve them by using tools which are as perfect as possible and concepts which are fit to be generalized.

[David Hilbert, International Congress of Mathematicians in Paris, 1900]

SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

PAR M. DAVID HILBERT (Göttingen),

TRADUITE PAR M. L. LAUGEL (1).

Qui ne soulèverait volontiers le voile qui nous cache l'avenir afin de jeter un coup d'œil sur les progrès de notre Science et les secrets de son développement ultérieur durant les siècles futurs? Dans ce champ si fécond et si vaste de la Science mathématique, quels seront les buts particuliers que tenteront d'atteindre les guides de la