Motivation

MICO model

Let $Q \subseteq \mathbb{R}^{n+d}$ be a compact convex set and $f : \mathbb{R}^{n+d} \rightarrow \mathbb{R}$ a convex function.

\[
\begin{align*}
\min & \quad f(x, y) \\
\text{s.t.} & \quad (x, y) \in Q, \\
& \quad x \in \mathbb{Z}^d, \ y \in \mathbb{R}^n.
\end{align*}
\]
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Let $Q \subseteq \mathbb{R}^{n+d}$ be a compact convex set and $f : \mathbb{R}^{n+d} \rightarrow \mathbb{R}$ a convex function.

$$\min f(x, y)$$

s.t. $(x, y) \in Q, x \in \mathbb{Z}^d, y \in \mathbb{R}^n$.

**Why study this model?**
- (MILP) and (CO) are about to become a technology.
- (MICO) seems to be the next natural step.
- Optimization over continuous relaxation is “tractable”.
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\end{align*}$$

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**What do we aim at?**

- Algorithmic schemes amenable to an analysis.
- Understand structural properties such as optimality conditions.
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\]

The central question

Can we follow algorithmic ideas in CO and adapt it to the mixed integer setting?

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For convex $g_i$ presented by a first order oracle, let

$$K = \{ x \mid g_i(x) \leq 0, \ \forall \ i \} \text{ and } Z = K \cap Z^n.$$
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\[ K = \{ x \mid g_i(x) \leq 0, \forall i \} \quad \text{and} \quad Z = K \cap \mathbb{Z}^n. \]

Cutting plane method, outer-approximation for $\min \{ c^T x \mid \text{s.t. } x \in Z \}$

Generate sequences of points $x_1, \ldots, x_l$ from linear integer relaxations:

\[
x_j = \arg \min c^T x \quad \text{s.t. } x \in \mathbb{Z}^n,
\]

\[
\nabla g_i(x_k)^T (x - x_k) \leq 0, \quad k < j
\]

References: Kelly '60, Westerlund, Pettersson, Duran, Viswanathan, Grossmann, Fletcher, Leyffer, Bonami et al.
For convex $g_i$ presented by a first order oracle, let

$$K = \{ x \mid g_i(x) \leq 0, \ \forall \ i \}.$$  

Theorem [Lenstra ’83] [Grötschel, Lovász, Schrijver ’88]

For any fixed $n \geq 1$, there exists an oracle-polynomial algorithm that decides whether $K \cap \mathbb{Z}^n = \emptyset$.

The engine:

- **Ellipsoid method:**
  - Grötschel, Lovász, Schrijver 86
  - Yudin, Nemirovskii 76
  - Khachiyan 96

- **Shortest lattice vector problem:**
  - Khinchin’s flatness theorem. [Kannan 83]

- **Closest lattice vector problem:**
  - Babai, Kannan 83

Nice surveys: [Eisenbrand ’08, Hildebrand and Köppe ’12]
The continuous case without constraints

**Theorem.**
Let $f$ be convex and continuously differentiable on its domain.
Let $x^* \in \text{dom } f$. Then, $x^*$ attains the value
$$\min\{f(x) \mid x \in \text{dom } f\}$$
if and only if
$$\nabla f(x^*) = 0.$$
The continuous case without constraints

**Theorem.** Let $f$ be convex and continuously differentiable on its domain. Let $x^* \in \text{dom } f$. Then, $x^*$ attains the value

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if and only if

$$\nabla f(x^*) = 0.$$

The unconstrained mixed integer case: [Baes, Oertel, W.]

**Theorem.** Let $f : \mathbb{R}^{n+d} \rightarrow \mathbb{R}$ be a continuous convex function. Then, $x^* \in \mathbb{Z}^n \times \mathbb{R}^d$ attains the value

$$\min\{f(x) | x \in \text{dom } f, x \in \mathbb{Z}^n \times \mathbb{R}^d\}$$

if and only if there exist $k \leq 2^n$ points $x_1 = x^*, x_2, \ldots, x_k \in \mathbb{Z}^n \times \mathbb{R}^d$ and vectors $h_i \in \partial f(x_i)$ such that the following conditions hold:

(a) $f(x_1) \leq \ldots \leq f(x_k)$,

(b) $\{x | h_i^T(x - x_i) < 0 \ \forall i\} \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \emptyset$,

(c) $h_i \in \mathbb{R}^n \times \{0\}^d$ for $i = 1, \ldots, k$. 
Proof. (assume $\text{dom } (f) = \mathbb{R}^{n+d}$ and $f$ is differentiable.

First direction

Given $x^*$ and $x_1 = x^*, x_2, \ldots, x_k \in \mathbb{Z}^n \times \mathbb{R}^d$ such that

(a) $f(x_1) \leq \ldots \leq f(x_k)$,
(b) $\{x \mid \nabla f(x_i)^T (x - x_i) < 0 \ \forall i\} \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \emptyset$,
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April 2015
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(c) \( \nabla f(x_i) \in \mathbb{R}^n \times \{ 0 \}^d \) for \( i = 1, \ldots, k \).

Then

\( \{ x \mid \nabla f(x_i)^T (x - x_i) < 0 \ \forall \ i \} \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \emptyset \).

Hence, for every \( \bar{x} \in \mathbb{Z}^n \times \mathbb{R}^d \) there exists \( x_i \) such that

\[ f(\bar{x}) - f(x_i) \geq \nabla f(x_i)^T (\bar{x} - x_i) \geq 0. \]

Then from (a), \( f(\bar{x}) \geq f(x_i) \geq f(x^*) \).
Proof continued.

**Converse direction**

Given $x^*$ optimal. Let $X^*$ be the set of all mixed integer optimal solutions. If there exists $\bar{x} \in X^*$ such that $0 \in \partial f(\bar{x})$, then the result follows.
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Otherwise, we define $F : \mathbb{R}^n \mapsto \mathbb{R}$, $F(z) = \min \{ f(z, y) \mid y \in \mathbb{R}^d \}$. $F$ is convex. Let $y_z$ attain value $F(z)$. Then, $\nabla f(z, y_z) = (\nabla F(z), 0)$. Let

$$L = \{ x \in \mathbb{R}^n \mid \nabla F(z)^T (x - z) < 0 \ \forall z \in \mathbb{Z}^n \}.$$
Proof continued.

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**Note that** $L \cap \mathbb{Z}^n = \emptyset$!

From [Doignon 73] there are $k \leq 2^n$ points $z_1, \ldots, z_k \in \mathbb{Z}^n$ such that

$$\bar{L} := \{x \in \mathbb{R}^n \mid \nabla F(z_i)^T(x - z_i) < 0 \ \forall i = 1, \ldots, k\}$$

satisfies $\bar{L} \cap \mathbb{Z}^n = \emptyset$. Wlog, $(z_1, y_1) = x^*$ and then together with $(z_1, y_{z_i})$ conditions (a) - (c) are satisfied.
Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $f(x) := \|Ax - c\|_2^2$ with

$$A := \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \text{ and } c := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

The continuous minimum of $f$ is attained at $(1/2, 1/2)^T$. Let

$$x_1 = (0, 0), \ x_2 = (0, 1), \ x_3 = (1, 0), \ x_4 = (1, 1).$$

$L = \{x \mid \nabla f(x_i)^T (x - x_i) \leq 0, \ \forall i\}.$
Example

Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $f(x) := \|Ax - c\|^2_2$ with

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$L = \{x \mid \nabla f(x_i)^T(x - x_i) \leq 0, \ \forall i\}$. We obtain

$$f(x_1) \leq f(x_i) \text{ for } i = 2, 3, 4, \\ \text{int}(L) \cap \mathbb{Z}^2 = \emptyset. \ \text{Therefore,} \ x_1 = \arg\min_{z \in \mathbb{Z}^2} f(z).$
KKT theorem under standard assumptions
Feasible $x^\star$ is optimal $\iff \exists h_f \in \partial f(x^\star), h_{g_i} \in \partial g_i(x^\star), \lambda_i \geq 0 \ \forall \ i, \ h_f + \sum_{i=1}^{m} \lambda_i h_{g_i} = 0 \ \text{and} \ \lambda_i g_i(x^\star) = 0 \ \forall i.$
A mixed integer KKT theorem [Baes, Oertel, W. 2014]

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$$h_f + \sum_{i=1}^{m} \lambda_i h_{g_i} = 0$$ and $\lambda_i g_i(x^\star) = 0 \; \forall \; i.$

For the mixed integer version and feasible $x^\star \in \mathbb{Z}^n \times \mathbb{R}^d$:

optimality $\iff \exists \; k \leq 2^n$ points $x_1 = x^\star, x_2, \ldots, x_k \in \mathbb{Z}^n \times \mathbb{R}^d$ and $k$ vectors $u_1, \ldots, u_k \in \mathbb{R}^{m+1}_+$ with $h_{i,m+1} \in \partial f(x_i)$, and $h_{i,j} \in \partial g_j(x_i) \; \forall j$ and

(a) If $g(x_i) \leq 0$ then $f(x_i) \geq f(x_1), \; u_{i,m+1} > 0$ and $u_{i,j} g_j(x_i) = 0 \; \forall j,$

(b) If $g(x_i) \nleq 0$ then $u_{i,m+1} = 0$ and $u_{i,k}(g_k(x_i) - g_l(x_i)) \geq 0 \; \forall k, l,$
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(d) $\sum_{j=1}^{m+1} u_{i,j} h_{i,j} \in \mathbb{R}^n \times \{0\}^d$ for $i = 1, \ldots, k$. 
A property of the certificate: integer projection property

The Euclidean Projection problem

Given $y \in \mathbb{R}^n$. Find

$$y^* = \text{argmin} \{ \|x - y\|_2 \mid g(x) \leq 0, x \in \mathbb{R}^n \}.$$ 

$y^*$ is uniquely determined and satisfies

for all feasible $x$, \quad \|x - y\|_2 \geq \|x - y^*\|_2.$
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The mixed integer version

For \( y \in \mathbb{R}^{n+d} \), the certificate \( x_1, \ldots, x_k \) for

\[
\min\{\|x - y\|_2 \mid g(x) \leq 0, x \in \mathbb{Z}^n \times \mathbb{R}^d \}
\]

satisfies the projection property:

for mixed-integer feasible \( x \) \( \exists \ 1 \leq i \leq k \) for which

\[
\|x - y\|_2 \geq \|x - x_i\|_2.
\]
From KKT to duality

Assumptions

Let $f : \mathbb{R}^{n+d} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^m$ be differentiable, convex functions, $\emptyset \neq \{ x \in \mathbb{R}^{n+d} | g(x) \leq 0 \} \subset \text{dom } f$ is compact. Let $g$ fulfill the (mixed-integer) Slater condition.

Continuous Lagrangian duality

$$f^* = \min_{x \in \mathbb{R}^n} \{ f(x) | g(x) \leq 0 \} = \max_{\alpha, u \in \mathbb{R}^m_+} \{ \alpha | \alpha \leq f(x) + u^T g(x) \forall x \in \mathbb{R}^n \}.$$
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\]

Mixed integer duality [Baes, Oertel, W. 2014]
\[
\begin{align*}
\min_{x \in \mathbb{Z}^n \times \mathbb{R}^d} & \{ f(x) | g(x) \leq 0 \} \\
= \max_{\alpha \in \mathbb{R}^2, U \in \mathbb{R}^{2^n \times m}} & \{ \alpha | \exists \pi : \mathbb{Z}^n \times \mathbb{R}^d \to \{1, \ldots, 2^n\} \text{ s.t.} \\
& \forall x \in \mathbb{Z}^n \times \mathbb{R}^d \alpha \leq f(x) + U_{\pi(x)} g(x) \text{ or } 1 \leq U_{\pi(x)} g(x) \}.
\end{align*}
\]
Any lattice free polyhedron provides us with a dual bound.

Consider $f$ convex.

Let $y^*$ be the continuous optimum. Let $L$ be any lattice free polyhedron such that

$$y^* \in \text{int} \ (L).$$

Then the integer optimum $x$ satisfies

$$f(x) \geq \min \{ f(z) \mid z \in \delta(L) \}.$$
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How to shrink a polytope so that it becomes lattice free?

**Shrinking of a polytope**

- For a polytope $P$, the centroid
  \[ c_P = \frac{\int_P x \, dx}{\text{vol}(G)}. \]
- $P_\lambda := \lambda(P - c_P) + c_P$.
- It seems difficult to determine a description of $P_\lambda$ from $P$ without knowing $c_P$. 
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Shrinking of a polytope

- For a polytope $P$, the centroid $c_P = \frac{\int_P x \, dx}{\text{vol}(G)}$.
- $P_\lambda := \lambda(P - c_P) + c_P$.
- It seems difficult to determine a description of $P_\lambda$ from $P$ without knowing $c_P$.

Task: Minimize $\lambda$ such that $P_\lambda \cap \mathbb{Z}^n \neq \emptyset$

leads to a mixed integer linear program in dimension $n + 1$:

$$t^* = \max t$$

$$a_i^T x + \omega(P, a_i)t \leq b_i \quad \forall i$$

$$x \in \mathbb{Z}^n, \quad t \geq 0.$$ 

$(x^*, t)$ feasible implies $x^* \in P_{1-t}$ and $x^* \in \{x \mid x + t(P - P) \subseteq P\}$. 

Robert Weismantel

April 2015
An algorithm

The setting

- Let $K$ be a convex set presented by a first order oracle.
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- Choose a constant $0 < \lambda < 1$. 
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The steps for testing $K \cap \mathbb{Z}^n = \emptyset$:

- **Step 1**: Let $P = \{x \mid Ax \leq b\}$ be a polytope containing $K$.

\[ P \cap \mathbb{Z}^n \neq \emptyset \]

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If $P \lambda \cap \mathbb{Z}^n = \emptyset$, generate subproblems.

If $x^* \notin K$, separate $x^*$. This can be accomplished by adding $\nabla g_j(x^*)^T(x - x^*) < 0$.
An algorithm

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The steps for testing $K \cap \mathbb{Z}^n = \emptyset$:

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- **Step 1**: Let $P = \{x \mid Ax \leq b\}$ be a polytope containing $K$.
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- **Step 3**: Let $x^* \in P_\lambda \cap \mathbb{Z}^n$.
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    \[ \nabla g_j(x^*)^T (x - x^*) < 0. \]
How to reconstruct a dual certificate from this algorithm?

The engine:

Let $G$ be a compact convex set, let $H$ be a halfspace and let $0 < \lambda < 1$. If $G_\lambda \cap H \neq \emptyset$, 

\[
\frac{\text{vol} (G \cap H)}{\text{vol} (G)} \geq (1 - \lambda)^n \left( \frac{n}{n + 1} \right)^n.
\]

This is an extension of a theorem of Grünbaum 1960 ($\lambda = 0$).
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Full dimensionality is lost

Iterations $k$ until $\text{vol}(P) \leq \frac{1}{n!}$:

$$k \leq \frac{n \left[ \log(2B) + \log(n) \right]}{(1 - \lambda)^n \left( \frac{n}{n+1} \right)^n}.$$
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**Next step:**

Apply the algorithm recursively to all lower dimensional problems.

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Next step:
Apply the algorithm recursively to all lower dimensional problems.

The certificate
Let $x^1, \ldots, x^k$ be all points with separating hyperplanes $c_i^T x \leq \gamma_i$ generated in the course of the algorithm.

$$L = \{x \in \mathbb{R}^n \mid c_i^T x < \gamma_i\}$$

is a lattice free polyhedron.

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