An Introduction to the Orbit Method

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September 11, 2012

Outline

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2. Coadjoint orbits
   - The coadjoint representation
   - The coadjoint orbits of SU(2)
3. Geometric Quantization
   - Axioms for prequantization
   - The integrality condition and the prequantization assignment
   - Kähler polarizations and the quantization assignment
4. The irreducible unitary representations of SU(2)
5. Conclusions

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  The Orbit Method is a method to determine all irreducible unitary representations of a Lie group.
- Why is it useful?
  Representation theory remains the method of choice for simplifying the physical analysis of systems possesing symmetry.
- The Orbit Method is entangled with its physical counterpart geometric quantization, which is an extension of the canonical quantization scheme to curved manifolds.

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Representations

Definition (*Representation*)

A representation of a group G on a vector space V is a group homomorphism from G to $GL(V)$, i.e. a map $\rho : G \to GL(V)$ such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$.

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- Loosely speaking: a representation makes an identification between abstract groups and more managable linear transformations of vector spaces.
- Reduce group-theoretic problems to problems in linear algebra, which is well-understood.

The adjoint and coadjoint representation

- Let G be a matrix group, i.e. a group of invertible matrices, and let V be its Lie algebra $\mathfrak{g}$. Then the adjoint representation Ad is defined by
$\mathrm{Ad}(g)X = gXg^{-1}$ for $g \in G$, $X \in \mathfrak{g}$,
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- Let G be a Lie group. The coadjoint representation Ad^* is the dual of the adjoint representation Ad, defined by
  $\mathrm{Ad}^*(g) \equiv \mathrm{Ad}(g^{-1})^* \quad (g \in G)$ with
  $\langle \mathrm{Ad}(g^{-1})^* F, X \rangle \equiv \langle F, \mathrm{Ad}(g^{-1})X \rangle$ for $X \in \mathfrak{g}$, $F \in \mathfrak{g}^*$.

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  $\langle \mathrm{Ad}(g^{-1})^* F, X\rangle \equiv \langle F, \mathrm{Ad}(g^{-1})X\rangle$ for $X \in \mathfrak{g}$, $F \in \mathfrak{g}^*$.
- In case G is a matrix group then $\mathfrak{g} \cong \mathfrak{g}^*$ and the coadjoint representation is just matrix conjugation again.

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- Hence coadjoint orbits are always even-dimensional!

The coadjoint orbits of SU(2): part I

- $\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$

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- Different coadjoint orbits of $SU(2)$ are characterized by different values for K.

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- In terms of the complex coordinates z and $\bar{z}$ (the stereographic coordinates) it locally equals
  $\omega = i\frac{1}{2R}\left(1 + \frac{z\bar{z}}{4R^2}\right)^{-2} dz \wedge d\bar{z}$.

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  Q : Classical observables $\to$ Quantum observables
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- **(Q4)**: Diracs quantum condition: $[Q(f), Q(g)] = -i\hbar Q(\{f, g\}) \;\; (f, g \in C^\infty(M, \mathbb{R}))$.

A Q satisfying axioms **(Q1) - (Q4)** and that puts into correspondence an operator on H to every classical observable $f \in C^\infty(M, \mathbb{R})$ is called a prequantization and the corresponding H is called the prequantum Hilbert space.

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- A prequantum Hilbert space consisting of square integrable scalar functions appears not to be the correct space to consider!
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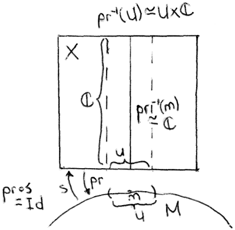
Definition (*Line bundle*)

A line bundle over a manifold M is a real manifold X (called the total space) together with a smooth map $\mathrm{pr} : X \to M$ (called the projection) such that

- $\mathrm{pr}^{-1}(m)$ is a copy of $\mathbb{C}$ for all $m \in M$.
- $\mathrm{pr}^{-1}(U) \cong U \times \mathbb{C}$ for a neighbourhood U of $m \in M$.

How is the prequantum Hilbert space build up?: part II



- A section of a line bundle X is a map $s : U \subset M \rightarrow X$ such that $\mathrm{pr} \circ s = I$.

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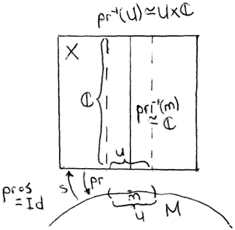

- A section of a line bundle X is a map $s : U \subset M \to X$ such that $\mathrm{pr} \circ s = I$.
- If we can define an inner product $\langle \cdot, \cdot \rangle$ (way of measuring) and connection ∇ (covariant way of differentiating sections) on X in a consistent manner then we call X a Hermitian line bundle with connection.

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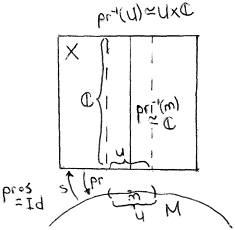

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How is the prequantum Hilbert space build up?: part III

- The curvature 2-form Ω on a Hermitian line bundle with connection is defined by
  $\Omega(X, Y) \equiv \frac{i}{2}([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})$
  for $X, Y \in \mathrm{Vect}(M)$.

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The integrality condition: part I

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Theorem

In order for a Hermitian line bundle with connection over (M, ω) with $\Omega = \frac{\omega}{\hbar}$ to exist the symplectic form ω on M needs to satisfy the Integrality Condition.

The integrality condition: part II

Integrality Condition

There exists an open cover $U = \{U_j\}$ of M such that the cohomology class defined by ω in $H^2(U, \mathbb{R})$ contains a cocycle $z \equiv \{z_{ijk}\}$ in which all z_{ijk}s are 2π multiples of integers.

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Idea of proof of the 'sufficience'-part of the Theorem

- Take a contractible open cover $U = \{U_j\}$ of M such that the symplectic form ω satisfies the Integrality Condition. Then there exist a real 1-form β such that
  $\omega = d\beta_j$ on U_j.
  Subsequently, there exists a smooth real anti-symmetric function f_{ij} such that
  $df_{ij} = \beta_i - \beta_j$ on $U_i \cap U_j$ and
  $a_{ijk} = f_{ij} + f_{jk} + f_{ki}$ constant on $U_i \cap U_j \cap U_k$.

The integrality condition: part III

- By the integrality condition, from this one can define constant integer-valued functions $z_{ijk} \equiv \tilde{f}_{ij} + \tilde{f}_{jk} + \tilde{f}_{ki}$ on $U_i \cap U_j \cap U_k$ which form a cocycle.

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- Then define $c_{ij} = \exp(i\tilde{f}_{ij})$. These are transition functions (i.e. they satisfy $c_{ij}c_{jk}c_{ki} = 1$, $c_{ij}c_{ji} = 1$).

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- Then define $c_{ij} = \exp(i\tilde{f}_{ij})$. These are transition functions (i.e. they satisfy $c_{ij}c_{jk}c_{ki} = 1$, $c_{ij}c_{ji} = 1$).
- From these transition functions one can then construct a Hermitian line bundle with connection with curvature $\Omega = \frac{d\beta}{\hbar} = \frac{\omega}{\hbar}$.

The integrality condition for $M = S^2$

- In case $M = S^2$ (the coadjoint orbits of $SU(2)$) the integrality condition states:
  $\int_{S^2} \omega \in 2\pi\mathbb{Z} \Leftrightarrow \int_{S^2} R \sin(\theta) d\theta d\phi \in 2\pi\mathbb{Z} \Leftrightarrow 4\pi R \in 2\pi\mathbb{Z} \Leftrightarrow R \in \mathbb{Z}/2.$

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- As a consequence spheres should be of half-integer radius for the corresponding prequantum Hilbert space to exist.

The prequantization assignment: part I

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- Prequantum Hilbert space: the space of all square-integrable sections $s : U \subset M \to X$.
- The prequantization assignment (satisfying only **(Q1)** - **(Q4)**) equals

$$Q(f) = -i\hbar\nabla_{X_f} + f,$$

for $f \in C^\infty(M, \mathbb{R})$, ∇ the connection and X_f the Hamiltonian vector field corresponding to f,
i.e. a vector field satisfying $2\omega(X_f, \cdot) = -df(\cdot)$.

The prequantization assignment: part II

- In case the symplectic manifold is a sphere of half-integer radius the prequantization assignment equals
$Q(f) = -i\hbar\left\langle \frac{1}{R^2 \sin(\theta)}\left(\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \phi} - \frac{\partial f}{\partial \phi}\frac{\partial}{\partial \theta}\right) + i\cot(\theta)\frac{\partial f}{\partial \theta}\right\rangle + f$
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in terms of the local coordinates θ and ϕ.
- In terms of the complex coordinates z and $\bar{z}$ it equals
$Q(f) = -i\hbar\left\langle 2R^2\left(1 + \frac{z\bar{z}}{4R^2}\right)^{-2}\left(\frac{\partial f}{\partial \bar{z}}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \bar{z}}\right) - \frac{4\bar{z}R^4}{(4R^2+z\bar{z})^3}\frac{\partial f}{\partial \bar{z}} \right\rangle + f$
for $f \in C^\infty(S^2, \mathbb{R})$.

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Polarizations: part I

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- How?
  Loosely speaking, by putting an additional structure on the symplectic manifold M by considering certain subspaces of its complexified tangent bundle $TM^{\mathbb{C}} \equiv \coprod_{m \in M} (T_m M)^{\mathbb{C}}$.

Polarizations: part II

- Example: in case the phase space is flat space $\mathbb{R}^{2n}$ equipped with the coordinates $(q^1, ..., q^n, p_1, ..., p_n)$, observables on the prequantum Hilbert space are functions depending on all these coordinates $(q^1, ..., q^n, p_1, ..., p_n)$, whereas observables on the Hilbert space are functions depending only on the coordinates $(q^1, ..., q^n)$ or $(p_1, ..., p_n)$.

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Definition (*Kähler manifold*)

A Kähler manifold is a complex manifold (M, J) which at the same time is a symplectic manifold (M, ω) such that the complex structure J ($J^2 = -I$) and the symplectic structure ω are compatible: $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in \mathrm{Vect}(M, \mathbb{R})$.

Kähler polarizations: part II

- In case M is a Kähler manifold its complexified tangent space at $m \in M$ can be split up into the subspaces $T_mM^{(0,1)}$ and $T_mM^{(1,0)}$, which are the eigenspaces corresponding to the eigenvalues $\pm i$ of the restriction of the complex structure $J_m : T_mM \to T_mM$.

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- The spaces $TM^{(0,1)} \equiv \coprod_{m \in M} (T_m M)^{(0,1)}$ and $TM^{(1,0)} \equiv \coprod_{m \in M} (T_m M)^{(1,0)}$ are Kähler polarizations of M.

The sphere is a Kähler manifold

- Locally, in terms of the complex coordinates z^k and $\bar{z}^k$, a Kähler form can be written as:
  $\omega = 2i\partial\bar{\partial}K$,
  with $\partial \equiv dz^k \frac{\partial}{\partial z^k}$, $\bar{\partial} \equiv d\bar{z}^k \frac{\partial}{\partial \bar{z}^k}$ and K the Kähler potential.

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- The sphere of radius R is a Kähler manifold with Kähler potential equal to
  $K = R \log\left(1 + \frac{z\bar{z}}{4R^2}\right)$.

Hilbert space

- Hilbert space: the space H of all polarized, square-integrable sections.

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- If M is a sphere of half-integer radius $k \in \mathbb{Z}/2$, then
  1. The Hilbert space consists of all polynomials of degree $n \leq 2k$ in the complex coordinate z.
  2. The quantization assignment equals
     $Q(f) = -i\hbar \left\langle \frac{1}{R^2 \sin(\theta)} \left(\frac{\partial f}{\partial \theta} \frac{\partial}{\partial \phi} - \frac{\partial f}{\partial \phi} \frac{\partial}{\partial \theta} \right) + i \cot(\theta) \frac{\partial f}{\partial \theta} \right\rangle + f$
     for $f \in C^\infty(S^2, \mathbb{R}; P) \subset C^\infty(S^2, \mathbb{R})$,
     the space of observables f preserving the polarization $P \equiv (TS^2)^{(0,1)}$, i.e. the f such that $Q(f) : H \to H$.

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- *Isaac Newton encrypted his discoveries in analysis in the form of an anagram that deciphers to the sentence, 'It is worthwhile to solve differential equations'. Accordingly, one can express the main idea behind the orbit method by saying 'It is worthwhile to study coadjoint orbits'.* -
A.A. Kirillov

The Orbit Method

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- The 2 ingredients used are:
  1. Coadjoint orbits,
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- Application: what are the irreducible unitary representations of SU(2)?

The irreducible unitary representations of SU(2): part I

Outline procedure:

- Consider the sphere S^2 of radius 1 and cover it by the charts $U_\pm = S^2/x_\pm$ with $x_\pm$ the north and southpole of the sphere. A parametrization of U_+ in terms of the complex coordinates z_+ and $\bar{z}_+$ is given by
$$x^1 = \frac{1}{2}\frac{z_+ + \bar{z}_+}{1+|z_+|^2}$$
$$x^2 = \frac{-i}{2}\frac{z_+ - \bar{z}_+}{1+|z_+|^2}$$
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where x^1, x^2 and x^3 are the standard coordinates on $\mathbb{R}^3$.

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where x^1, x^2 and x^3 are the standard coordinates on $\mathbb{R}^3$.
- Now consider a sphere of radius $k = \lambda/2$ and note that $\omega_k = k\omega_1$. Then quantizing the coordinate functions x^1, x^2, x^3 in terms of z_+ and $\bar{z}_+$ gives the operators $Q(x^1) \equiv z_+^2 \frac{\partial}{\partial z_+} - 2\lambda z_+$, $Q(x^2) \equiv -\frac{\partial}{\partial z_+}$ and $Q(x^3) \equiv z_+ \frac{\partial}{\partial z_+} - \lambda$.

- The operators $Q(x^1)$, $Q(x^2)$, $Q(x^3)$ satisfy the same bracket relations as the generators of $\mathfrak{su}(2)$, i.e. $[X_1, X_2] = 2X_3, \;\; [X_1, X_3] = -2X_2, \;\; [X_2, X_3] = 2X_1$.

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- Hence they form a representation of $\mathfrak{su}(2)$ in the space of homogeneous polynomials in z of degree $n \leq 2k$, since this is an invariant subspace of the the space of all polynomials of degree $n \leq 2k$, which is the Hilbert space for the spheres of half-integer radius.

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- By lifting the information from the Lie algebra to the Lie group this gives all the irreducible unitary representations of $SU(2)$.

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Conclusions: part II

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- **Outlook**: using the Orbit Method to find all irreducible unitary representations of non-compact Lie groups, such as $SL(2, \mathbb{R})$ or the Virasoro group!

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