ARC STRUCTURE OF SINGULARITIES

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Introduction. This paper is motivated as much by some interesting possible truths which were encountered as by the things which could be established. Perhaps others will be able to complete the picture with the solutions of these implicit problems.

For simplicity we confine ourselves to the complex numbers. We consider abstract (algebraic) varieties over the complex numbers. One could also extend the results to complex analytic varieties if one had the resolution of singularities in this case.

Arcs are small portions of algebroid curves lying in the variety V. They are represented by convergent power series in a complex parameter t giving the coordinates $x(t)$ of a point of V. We consider the nonsingular arcs where $x(t)$ is nonsingular on V for general t. Associated with any algebraic subset W of the variety, there are the arcs where $x(0)$ is a point of W. From the existence of a resolution of the singularities of V, we find that the set of arcs associated with W decomposes into a finite number of families. The number of these is not more than the number of components of the image of W in a resolution. However, the families exist independently of any particular resolution.

For surfaces it seems possible that there are exactly as many families of arcs associated with a point as there are components of the image of the point in the minimal resolution of the singularities of the surface. This is the first open question.

In higher dimensions, the arc families associated with the singular set correspond to "essential components" which must appear in the image of the singular set in all resolutions. We do not know how complete is the representation of essential components by arc families.

Exemplifying essential components is a large class characterized by a condition on their birational types. A hypersurface appearing in a resolution as a component of the image of the singular set which is not birationally equivalent to a ruled variety is essential. It must appear in any resolution.

As illustrations, we present a few examples of singularities and their arc families and essential components.

Arc families. Let W be an algebraic subset of a variety V, and let $x_1, \ldots, x_N$ be locally valid coordinates corresponding to an affine space A^N. We consider

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parametrized arcs $x_i(t) = \sum_{\alpha=0}^{\infty} x_{i\alpha} t^\alpha$ where the series converge for t small, $\{x_{i0}\} = \{x_i(0)\}$ is a point of W, and $x(t)$ is a nonsingular point of V for general t. We study finite truncations $\{x_{i\alpha} | \alpha \leqslant a\}$ of the sets of coefficients. It turns out that the set of possible $x_{i\alpha}$ values for $\alpha \leqslant a$ is a constructible set. A constructible set is a finite union of sets, each of which is formed by a variety from which a finite number of subvarieties have been removed. (See [1].) In our case, we have an abstract constructible set related to various systems of local coordinates $x_{i\alpha}$. If $x_{i\alpha}$ and $\bar{x}_{i\alpha}$ are two such systems deriving from two local systems x_i and $\bar{x}_i$ for V, we see that they relate biholomorphically on this set of truncations. Because $\bar{x}(t) = \bar{x}(x(t))$ and $\bar{x}(x)$ is holomorphic, the coefficients $\bar{x}_{i\alpha}$ of the series for $\bar{x}(t)$ for $\alpha \leqslant a$ depend holomorphically on those for $x(t)$ for $\alpha \leqslant a$ by the chain rule for differentiation. Note that the derivatives of $\bar{x}(x)$ with respect to x at $t = 0$ become functions of the x_{i0}.

The closure of the set of a-truncations (possible values of the $x_{i\alpha}$ for $\alpha \leqslant a$) is composed of a certain number of varieties. We find that this number is constant for $a \gg 0$. A general point in a component of the closure of the set of a-truncations is necessarily a point of the constructible set of a-truncations.

When $a_2 > a_1$, the set of a_2-truncations projects into the set of a_1-truncations by ignoring the latter coefficients. For a_1 and a_2 large enough, each component of the closure of the a_2-truncation set will project into a dense subset of a component of the closure of the a_1-truncation set, because the number of these components is constant for $a \gg 0$. We may identify corresponding components of this sort for large a and call them $C_\lambda(a)$, defined for $a \geqslant a_o$, $\lambda = 1, 2, \ldots, \Lambda$. We define a family F_λ to consist of those arcs having an a-truncation in $C_\lambda(a)$ for all $a \geqslant a_o$. These families are not disjoint. In principle it might be difficult to compute the structure of the families; however, our examples will show that in a specific case they are often quite simple to describe.

Lifting of arcs. Let V^*, where $V^* \to V$ is birational, holomorphic, and proper, be any resolution of the singularities of V. We refer to the results of Hironaka for the existence of a resolution. His paper [2] also gives references to the other literature on the resolution of singularities. We find that each arc of the sort we consider is the holomorphic image under $V^* \to V$ of at least one arc in V^*. If the arc lies outside the set where $V \to V^*$ is not holomorphic, its image in V^* is unique. For this case, let $x(t)$ be the arc in V and y local coordinates in V^*. Then $y(x(t))$ is uniquely defined for $t \neq 0$ and is meromorphic. Since the image in V^* of a compact neighborhood of $x(0)$ is compact, there must exist a limit set corresponding to limits of $y(x(t)) = y(t)$ when $t \to 0$. Using local coordinates y suitable near a point of that limit set, we have meromorphic functions $y(t)$ which do not approach infinity as $t \to 0$. Therefore, these must be holomorphic so that $y(t)$ is an arc in V^*.

In the other case where $V \to V^*$ is not holomorphic on the arc $x(t)$, we can employ another resolution V_b where $V \to V_b$ is biholomorphic outside the singular set of V. (The resolutions of Hironaka are of this sort.) Since $x(t)$ lies outside the singular set of V, there is a unique corresponding arc $\xi(t)$ in V_b. Because V_b is nonsingular, we can extend $\xi(t)$ in a general way to a holomorphic vector function $\xi(t, u)$ lying in V_b such that $\xi(t, 0) = \xi(t)$, and for small t and u the induced transformation $V_b \to V^*$ is holomorphic at $\xi(t, u)$ except for $u = 0$. This is because the set where $V_b \to V^*$ is not holomorphic has codimension at least two. Any coordinate y_m of V^* now becomes a meromorphic function of t and u, and we can write $y_m = \psi(t, u)/\varphi(t, u)$ as the quotient of holomorphic functions. If $\varphi(t, u)$ vanishes on $u = 0$ to a higher order than $\psi(t, u)$, then for almost all t values $\lim_{u\to 0} y_m(t, u)$ does not exist. But if $\psi(t, u)$ vanishes as highly as $\varphi(t, u)$ on $u = 0$, then $\lim_{u\to 0} y_m(t, u)$ exists for almost all t values and is a meromorphic function of t. Now, since for each fixed value of t we have an arc $\xi(t, u)$ parametrized by u which must transform to an arc in V^*, we see that there must exist a local set of coordinates y for V^* such that $\lim_{u\to 0} y(\xi(t, u))$ is defined for almost all t values and is meromorphic. Then by the same argument as for the simpler case, we see that with proper coordinates, $y(t) = \lim_{u\to 0} y(\xi(t, u))$ will be holomorphic in t. This gives an arc $y(t)$ in V^* which corresponds with the arcs $\xi(t)$ in V_b and $x(t)$ in V.

For this lifting we have used quite essentially the one-dimensionalness of the arcs.

Finiteness and structure of the families. Let ${W^*}_1, \ldots, {W^*}_L$ be the components of the image of W in the resolution V^*. We find that there are not more than L families of arcs associated with W, i.e., $\Lambda \leqslant L$. An arc $x(t)$ where $x(0)$ is on W will correspond to an arc $y(t)$ in V^* where $y(0)$ is on some ${W^*}_j$. Because $V^* \to V$ is holomorphic, the coefficients of an a-truncation of $x(t)$ are determined, holomorphically and rationally, by those of an a-truncation of $y(t)$. So the set of the a-truncations of arcs associated with W is the image under a rational holomorphic mapping of the sets of the a-truncations of the arcs associated with the ${W^*}_j$. We can show that these sets are varieties, and it follows that the a-truncations of arcs associated with W form a constructible set whose closure has at most L components.

To study the structure of the a-truncations of arcs associated with ${W^*}_j$, we can employ local coordinates $\theta_1, \ldots, \theta_r$ (where $r = \dim V^* = \dim V$) in V^* valid near a point of ${W^*}_j$ which are algebraic functions of the coordinates y. Expressed in these, an arc $y(t)$ becomes $\theta(t) = \theta_p(t) = \sum_{\beta=0}^{\infty} \theta_{p\beta} t^{\beta}$. Since the transformations between the coordinates y and θ are holomorphic, the a-truncations of $y(t)$ and $\theta(t)$ are algebraic holomorphic functions of each other. The coefficients $\theta_{p\beta}$, $\beta \leqslant a$ are unrestricted except that $\{\theta_{po}\}$ must be a point on ${W^*}_j$. Thus the a-truncations of arcs $\theta(t)$ form an algebraic set locally equivalent to the product of ${W^*}_j$ with an affine ar-space (Note that ${W^*}_j$ might be reducible in the local coordinates θ.) By the correspondence, the a-truncations of $y(t)$ also form an algebraic set locally isomorphic to the product of ${W^*}_j$ with affine ar-space. But since ${W^*}_j$ is irreducible, this must be a variety (irreducible). Actually this variety has the form that the $(a + 1)$-truncations are a ruled variety with an affine r-space as fiber over the

a-truncations as base and the 0-truncations are W^*_j. Here the fibers correspond to the tangent planes to V^*. Note incidentally that, corresponding to any a-truncation of arcs at W^*_j, there are arcs which lie outside the image of the singular set of V because the coefficients $\theta_{p\beta}$, $\beta > a$ may be chosen arbitrarily except for the convergence requirement. Thus each such truncation maps into a truncation in V corresponding to arcs of the sort we consider.

If $D_j(a)$ is the variety of a-truncations associated with W^*_j, then $D_1(a) \cup \cdots \cup D_L(a)$ maps holomorphically and rationally onto the constructible set of all a-truncations of arcs associated with W and into its closure $C_1(a) \cup \cdots \cup C_\Lambda(a)$, and we see that $\Lambda \leqslant L$.

Correspondence of families to components. We want to show that for $a \gg 0$ each $C_\lambda(a)$ corresponds in a natural way with a specific W^*_j. In this connection, we find that for $a \gg 0$, a general member of any $C_\lambda(a)$ corresponds only to arcs which in V^* pass through a specific point (i.e., for a corresponding arc $y(t)$ in V^*, $y(0)$ is determined).

For each pair of coordinate systems x and y for affine portions of V and V^*, we may choose specific rational functions $y(x)$ to represent the transformation $V \to V^*$. Let S be the set in the ambient affine space of x where $y(x)$ is not holomorphic. S will not contain V and we can call S_v the intersection $S \cap V$.

Take a general member of $C_\lambda(a_1)$. This will be the image under the induced mapping of truncations of a general member of some $D_j(a_1)$ to which will correspond an arc $y^{(1)}(t)$ passing through a general point of W^*_j (for $t = 0$) in a general direction. Being thus general $y^{(1)}(t)$ will not lie in the image of the set S_v, and the corresponding arc $x^{(1)}(t)$ in V will not lie in S (considering the sets S and S_v related to these local coordinates x and y). Suppose the distance of $x^{(1)}(t)$ from S is $\geqslant C_1|t|^p$. Now for $a > p$ the distance from S, modulo $|t|^{(p+a)/2}$, say, of arcs corresponding to an a-truncation is determined by the truncation. A general member of $C_\lambda(a)$ must have this distance $\geqslant C|t|^p$ since a particular member has distance $\geqslant C_1|t|^p$.

The first derivatives of the rational functions $y(x)$ will be finite away from S, and we can assume they are all $\leqslant C'd^{-\sigma}$ in absolute value, where d is the distance to S, in every compact neighborhood in the affine space of the coordinates x. Now assume $a \geqslant p\sigma$ and consider two arcs $x(t)$ and $\hat{x}(t)$ whose a-truncation is the same and is general in $C_\lambda(a)$. We have $|x - \hat{x}| \leqslant C''|t|^{a+1}$, and for the corresponding arcs $y(t)$ and $\hat{y}(t)$ in V^*, we have $|y - \hat{y}| \leqslant BC''|t|^{a+1}C'(C|t|^p)^{-\sigma} \leqslant C'''|t|^{a-p\sigma+1}$ by the bounds, from which it follows that $y(0) = \hat{y}(0)$. (B is a constant depending on the number of coordinates y.) The a-truncation in $C_\lambda(a)$, being general, is the image of a general a-truncation from some $D_j(a)$. Suppose $y(t)$ corresponds to this a-truncation from $D_j(a)$; then $y(0)$ is general on W^*_j, and any arc $\hat{x}(t)$ corresponding to the given member of $C_\lambda(a)$ becomes an arc $\hat{y}(t)$ in V^* passing through this point $y(0)$ on W^*_j. So we see that for $a \gg 0$ a general member of $C_\lambda(a)$ corresponds to a specific general point on some W^*_j, and thus we obtain a pairing of the families F_λ with some subset of the W^*_j.

Now consider two different resolutions V^* and V^{**}. For $a \gg 0$, a general member of $C_\lambda(a)$ will correspond to arcs associated with general points of components ${W^*}_j$ and ${W^{**}}_k$ of the images of W in V^* and in V^{**}. Dilate ${W^*}_j$ and ${W^{**}}_k$ to codimension one by monoidal transformations, obtaining ${\overline{W}^*}_j$ and ${\overline{W}^{**}}_k$, and then resolve any resulting singularities to obtain new resolutions $\overline{V}^*$ and $\overline{V}^{**}$. From our previous conclusions, a general truncation in $C_\lambda(a)$ for $a \gg 0$ must correspond to arcs passing through a specific general point of some component of the image of W in $\overline{V}^*$ (or $\overline{V}^{**}$). But these arcs must project to arcs passing through a general point of ${W^*}_j$ (or ${W^{**}}_k$) in V^* (or V^{**}). Therefore, they must pass through specific general points of ${\overline{W}^*}_j$ and ${\overline{W}^{**}}_k$. The birational correspondence $\overline{V}^* \to \overline{V}^{**}$ will be holomorphic at a general point of ${\overline{W}^*}_j$, and because of the corresponding arcs we see that this point must transform into a general point of ${\overline{W}^{**}}_k$. This applies in both directions and it follows that ${\overline{W}^*}_j$ and ${\overline{W}^{**}}_k$ are in (induced) birational correspondence.

This shows that the components of the image of W in a resolution which correspond to the arc families F_λ are absolute and must appear in any resolution. The concept of equivalence is the induced birational correspondence of their monoidal transforms.

Essential components. We call these necessary components of the image of W in a resolution essential components associated with W, and we are particularly interested in certain cases. If W is the singular set of V, we call them simply essential components. If $V \to V^*$ is holomorphic outside the singular set, then the essential components will be components of the set in V^* where $V^* \leftrightarrow V$ is not biholomorphic. And if V is normal, they will be exceptional subvarieties of V^*. There will be at least one essential component for each component of the singular set of V.

Looking for examples of essential components, one finds a class described in birational terms. If a component of the image of the singular set of V in a resolution V^* is a hypersurface which is not birationally equivalent to a ruled variety, then it is essential. Actually, we can simply show that such a hypersurface must appear (dilated) in any nonsingular model of V. This implies that in a resolution it must be a component of the image of the singular set.

Let C be the hypersurface in V^*, and let $V^\#$ be another nonsingular model of V. The induced transformation $V^* \to V^\#$ is holomorphic at a general point of C because V^* is nonsingular and C has codimension one. The general points of C transform into a subvariety D of $V^\#$. For contradiction, we suppose D has codimension > 1, and show that it can then be transformed into a ruled variety birationally equivalent to C. Using results of Hironaka [2] we can apply monoidal transformations acting at the singularities of D until D is nonsingular, and then tranform D to codimension one, always keeping the ambient variety nonsingular. This gives us a variety $V_1^\#$ in which the general points of C transform either into the dilated D or into a subvariety D_1 of D. In this way we obtain a sequence $V^\#$, D; $V_1^\#, D_1; \ldots$.

Now consider the Jacobian of the transformation $V^* \to V_n$ near a general point of C using suitable local parameters in V^* and $V_n^\#$. Until D_n has codimension one, this must vanish to a certain order at that point of C. The transformations desingularizing D_n do not change this order of vanishing since they do not affect general points of D_n, but when we dilate D_n to codimension one, the order decreases, since the new Jacobian is the product of those for $V^* \to \bar{V}_n$ and $\bar{V}_n \to V_{n+1}$, where $\bar{V}_n$ is the variety obtained by desingularizing D_n in $V_n^\#$. Since the order cannot decrease indefinitely, we reach a hypersurface D_n in V_n which corresponds birationally with C and is a ruled variety, because it is the monoidal transform of a nonsingular subvariety of a nonsingular variety.

Here a question arises. Is there always a corresponding arc family for an essential component of this class? And, in general, how completely do the essential components correspond to the arc families?

One finds by examples that not all singularities can be resolved entirely by essential components. But it may be possible to resolve in steps such that the components introduced at each step satisfy suitable essentiality conditions. A canonical stepwise resolution process might proceed by dilating, at each step, an appropriate set of essential components in a standard manner.

Examples. The singularity $x^2 + y^2 + z^{n+1} = 0$ or $\xi\eta = z^{n+1}$ resolves minimally into n exceptional curves in a chain formation. Here it is easy to find the arc families, which are represented by arcs of the form $\xi = \alpha t^\nu + \cdots, \eta = \beta t^{n+1-\nu} + \cdots$, $z = \gamma t + \cdots$, with $\nu = 1, 2, \ldots, n$ and $\alpha\beta = \gamma^{n+1}$. An n-truncation is sufficient to distinguish all the arc families. With $x^2 + y^3 + z^6 = 0$, one finds a single arc family $x = \alpha t^3 + \cdots$, $y = \beta t^2 + \cdots$, $z = \gamma t + \cdots$. These arcs in general all have a certain direction at the singular point, being tangent to the line $x = y = 0$. Thus they do not correspond to the first exceptional curve obtained in resolution by quadratic transformations. The quadratic resolution produces two curves, of which the second corresponds to the arc family and the first is inessential. With $x^3 + y^3 + z^2 = 0$, one first finds three arc families of the sort $x = \alpha t + \cdots$, $y = \alpha\theta t + \cdots$, $z = \beta t^2 + \cdots$, where $1 + \theta^3 = 0$, which have special directions at the origin and do not correspond to the first quadratic transform. However, a closer analysis reveals another family of the form $x = \alpha t^2 + \cdots$, $y = \beta t^2 + \cdots$, $z = \gamma t^3 + \cdots$, and all four exceptional curves of the minimal resolution are represented.

The surface $z^2 = xy^2$ can be resolved by normalization, introducing $u = z/y$. Observe that the special arc $x = t$, $y = z = 0$, which lies in the singular set, does not lift to any arc in the resolved surface. Arcs of the sort $x = \alpha t^2 + \cdots$, $y = \beta t + \cdots$, $z = \gamma t^2 + \cdots$ constitute the family at the origin and correspond to the point $u = x = y = z = 0$ in the resolution.

The singularity $w^2 + x^2 + y^2 + z^m = 0$ resembles $x^2 + y^2 + z^{n+1} = 0$ in that it can be resolved by a sequence of quadratic transformations with the same sort of singularity appearing at each step. Presumably, this would be the canonical resolution for this singularity, in $[m/2]$ exceptional surfaces. However, most of these surfaces do not correspond to arc families and are not essential for the resolution of the singularity. When m is even, there are two ways of resolving the singularity by a single exceptional curve. Therefore, there is only one arc family and one essential component, and these correspond to the first surface of the quadratic resolution.

For odd or even m, one can rewrite $w^2 + x^2 + y^2 + z^m = 0$ as $uv + \eta^2 + z^m = 0$ in a continuum of different ways. Then a monoidal transformation centered at the line $v = \eta = z = 0$ produces two exceptional surfaces derived from the line and from the singular point. The surface over the point corresponds to the first surface of the quadratic resolution, which is an essential component. There is a remaining singularity $u\bar{v} + \bar{\eta}^2 z + z^{m-1} = 0$ where $\bar{v} = v/z$, $\bar{\eta} = \eta/z$. By introducing $\bar{\bar{v}} = \bar{v}/z$ and $\bar{\bar{v}}^{-1}$, this is transformed into a generally nonsingular curve, and another singular point remains. The curve corresponds to the second surface of the quadratic resolution. Near the singular point ($u = \bar{\bar{v}} = \bar{\eta} = z = 0$), we have $u\bar{\bar{v}} + \bar{\eta}^2 + z^{m-2} = 0$, which is a singularity of the original type and can be resolved by a sequence of quadratic transformations. But the surfaces obtained do not correspond to those of the quadratic resolution of the original point, since near their general points they have u comparable in size to $\bar{\bar{v}}$ or v comparable to z^2u instead of u and v comparable. From this it appears that for m odd there are at most two arc families and at most two essential components. These resolutions seem to be relatively minimal when m is odd. One could construct similar transformations holomorphic away from the singularity by first making a quadratic transformation, after which the center for the monoidal transformation would lie in the image of the original singular point. Observe also that in the stepwise quadratic resolution of this singularity, the surface dilated at each stage is essential, relative to the immediately preceding singularity.

The singular three-fold $zw + x^2y^2 = 0$ has two intersecting nodal curves. Corresponding to the singular set, there are two arc families and two essential components, both relating to the singular curves. Associated with the intersection point there are three arc families: $x = O(t)$, $y = O(t)$, $z = O(t^\nu)$, $w = O(t^{4-\nu})$, $\nu = 1, 2, 3$. Resolutions are possible using only the essential components, and in these the origin transforms into three curves. But for a symmetric resolution, it seems that at least two of the arc families at the intersection must correspond to surfaces.

With $xyz + w^2 = 0$, there are three intersecting nodal curves and three arc families for the singular set. At the intersection, $x = O(y)$, $y = O(t)$, $z = O(t^2)$ is typical among the three families there. One may resolve by monoidally transforming the curves one by one or by the ideal (xy, yz, zx, w), which monoidally transforms the complete singular set. In all these cases, the three families at the origin correspond to three curves variously arranged. These resolutions introduce only essential components, and the symmetric one looks like a nice canonical resolution for this variety.

Summary. For the convenience of the reader, we present here the main results.

PROPOSITION 1. *Corresponding to any algebraic subset W of a variety, there are a finite number of families of associated arcs $x(t)$ where $x(0)$ is on W.*

PROPOSITION 2. *Given a resolution of the variety, each arc family will correspond to a specific component of the image of W in the resolution.*

COROLLARY. *There are essential components for W which must appear as components of the image of W in any resolution, equivalence of components being the birational correspondence of their monoidal transforms.*

OBSERVATION. *If, in a resolution, a component of the image of the singular set of the singular variety is a hypersurface not birationally equivalent to a ruled variety, then it is an essential component and will appear in every resolution.*

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