Von Neumann-Morgenstern stable sets in the assignment market

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Outline

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1. Core and stable sets
2. Some weaknesses of the core
3. The assignment game
4. The compatible subgames
5. The stable set

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Cooperative TU games

A cooperative game with transferable utility is (N, v), where

- $N = \{1, 2, \ldots, n\}$ is the set of players and
- $$\begin{array}{rcl} v : 2^N & \longrightarrow & \mathbb{R} \\ S & \mapsto & v(S) \end{array}$$ is the characteristic function.

An imputation is a payoff vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$ that is

- Efficient: $\sum_{i \in N} x_i = v(N)$
- Individually rational: $x_i \geq v(i)$ for all $i \in N$.

Let $I(v)$ be the set of imputations of (N, v) and $I^*(v)$ be the set of preimputations (efficient payoff vectors).

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The dominance relation and the core

Let it be (N, v) and $x, y \in I^*(v)$:

- y dominates x via coalition $S \neq \emptyset$ $(y\, dom_S^v x) \Leftrightarrow x_i < y_i$ for all $i \in S$ and $\sum_{i \in S} y_i \leq v(S)$.
- y dominates x $(y\, dom^v x)$ if $y\, dom_S^v x$ for some $S \subseteq N$.

Definition (Gillies, 1959)

The core $C(v)$ of (N, v) is the set of preimputations undominated by another preimputation.

- If $C(v) \neq \emptyset$, then it coincides with the set of imputations undominated by another imputation.
- Equivalently, $C(v) = \{x \in I(v) \mid \sum_{i \in S} x_i \geq v(S),\ \text{for all } S \subseteq N\}$.

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The stable sets or von Neumann-Morgenstern solutions

Definition (von Neumann and Morgenstern, 1944)

Given (N, v), a subset of imputations $V \subseteq I(v)$ is a stable set if

1. two imputations $x, y \in V$ do not dominate one another (internal stability) and
2. any $y \in I(v) \setminus V$ is dominated by some $x \in V$ (external stability).

- The core is always included in any stable set.
- There are games with no stable set (Lucas, 1968, 1969).

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Some weaknesses of the core

The definition of the core of the game as the set of undominated outcomes is subject to the following conceptual query. Suppose we think of outcomes in the core as "good" or "stable". Then we should not exclude an outcome y just because it is dominated by some *other outcome; we should demand that the dominating outcome x itself be "stable". Otherwise, the argument for excluding y is rather weak and proponents of y can argue that replacing it with x would not lead to a more stable situation, so we may as well stay where we are.*

R. Aumann (1987) What is game theory trying to accomplish?

The glove-market game

- $N = M \cup M'$: each agent in M has a left-hand glove and each agent in M' has a right-hand glove.
- A glove alone is worthless. A left-right pair is worth 1.
- This game is $v(S) = \min\{|S \cap M|, |S \cap M'|\}$.
- If $|M| < |M'|$,
  $C(v) = \{x \in \mathbb{R}^N \mid x_i = 1 \text{ if } i \in M, x_i = 0 \text{ if } i \in M'\}$.
- Let it be $M = \{1\}$ and $M' = \{2, 3\}$, then $C(v) = \{(1, 0, 0)\}$.
- The core is based on what a coalition can do, not what it can prevent: in the glove-market game, the large side of the market can prevent any profit.
- $V = [(1, 0, 0), (0, 1, 0)]$ and $V' = [(1, 0, 0), (0, 0, 1)]$ are stable.
- Shapley (1959) proves the existence of (infinitely many) stable sets for the glove-market games.

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The assignment game (Shapley and Shubik, 1972)

- The assignment game is a cooperative model for a two-sided market (Shapley and Shubik, 1972).
- A good is traded in indivisible units (side-payments allowed).
- Each buyer in $M = \{1, 2, \ldots, m\}$ demands one unit and each seller in $M' = \{1, 2, \ldots, m'\}$ supplies one unit.
- Buyer i and seller j make a join profit of a_{ij} if they trade.

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1m'} \\ a_{21} & a_{22} & \ldots & a_{2m'} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \cdots & a_{mm'} \end{pmatrix}$$

- The cooperative game is defined by $(M \cup M', w_A)$, the characteristic function w_A being (for all $S \subseteq M$ and $T \subseteq M'$)

$$w_A(S \cup T) = \max\{ \sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\},$$

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The core: lattice structure

- ✓ Assignment games have a non-empty core.
- ✓ Given $\mu \in \mathcal{M}^*_A(M, M')$: $(u, v) \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+$ is in the core $\Leftrightarrow$

$$u_i + v_j \geq a_{ij} \text{ for all } (i, j) \in M \times M',$$
$$u_i + v_j = a_{ij} \text{ for all } (i, j) \in \mu,$$
$$u_i = 0 \text{ and } v_j = 0 \text{ if } i \text{ and } j \text{ are unmatched by } \mu.$$

- ✓ Inside the core third-party payments are excluded.
- ✓ $C(w_A)$ with the following partial order has a lattice structure:

$$(u, v) \leq_M (u', v') \Leftrightarrow u_i \leq u'_i, \quad \forall i \in M.$$

Let $(M \cup M', w_A)$ be an assignment market and (u, v), (u', v') two elements in $C(w_A)$. Then,

$$\left((\max\{u_i, u'_i\})_{i \in M}, (\min\{v_j, v'_j\})_{j \in M'}\right) \in C(w_A) \text{ and}$$
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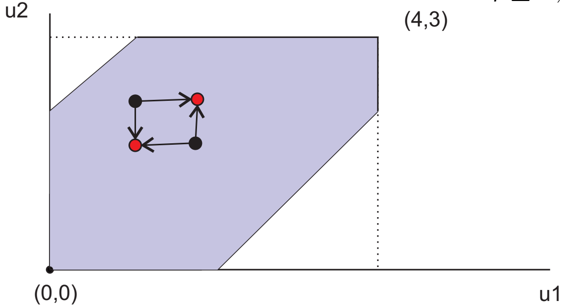
Lattice structure of the core

	3	4
1	**4**	1
2	2	**3**

$$u_1 + v_3 = 4$$
$$u_1 + v_4 \geq 1$$
$$u_2 + v_3 \geq 2$$
$$u_2 + v_4 = 3$$
$$u_i \geq 0, \; v_j \geq 0.$$



- $(\overline{u}, \underline{v})$ and $(\underline{u}, \overline{v})$, optimal core points for each side.
- $(\overline{u}, \underline{v}) = (4, 3; 0, 0)$, $(\underline{u}, \overline{v}) = (0, 0, 4, 3)$.

Core stability

Definition (Solymosi and Raghavan, 2001)

An assignment game $(M \cup M', w_A)$ with as many buyers as sellers has a dominant diagonal if for any optimal matching μ and all $k \in M$,

$$a_{k\mu(k)} \geq \max\{a_{kj}, a_{i\mu(k)}\}, \text{ for all } j \in M' \text{ and } i \in M\,.$$

This property is equivalent to saying that each agent has a null minimum core payoff.

Theorem (Solymosi and Raghavan, 2001)

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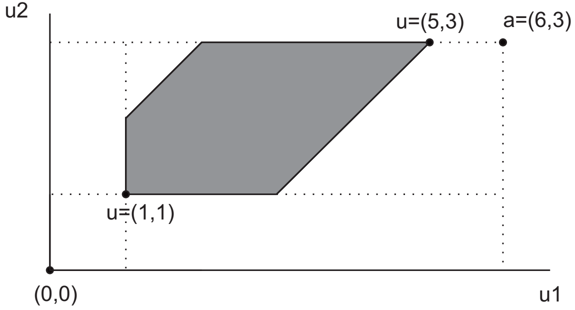
Example 2

	1'	2'	3'
1	**6**	2	1
2	4	**3**	1

✓ $(\overline{u}, \underline{v}) = (5, 3; 1, 0, 0)$.

✓ $(\underline{u}, \overline{v}) = (1, 1; 5, 2, 0)$.

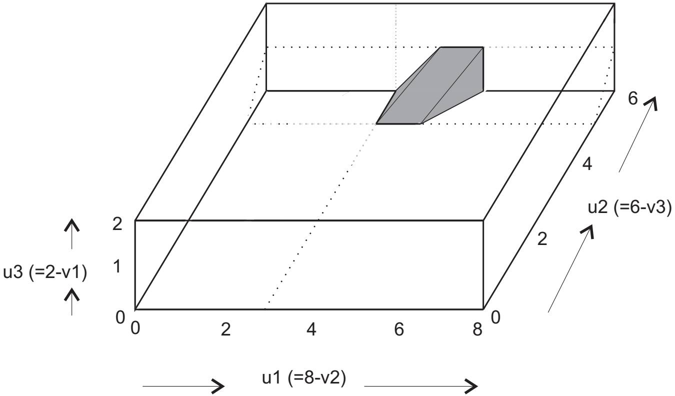
✓ This core is not stable.



Example 3: Shapley and Shubik, 1972

	1'	2'	3'
1	5	**8**	2
2	7	9	**6**
3	**2**	3	0

- ✓ Optimal matching: $\mu = \{(1, 2'), (2, 3'), (3, 1')\}$.
- ✓ $(\overline{u}, \underline{v}) = (5, 6, 1; 1, 3, 0)$, $(\underline{u}, \overline{v}) = (3, 5, 0; 2, 5, 1)$.
- ✓ This core is not stable.



Do there exist stable sets for the assignment game?

- In Shapley and Shubik (IJGT, 1972), "*The Assignment Game I: The Core*", the authors end: "*It may not be possible to realize the bargaining potentials described above within a given institutional form... it behoves to us to explore and correlate a number of different solution concepts. This we hope to do in subsequent papers.*"
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The compatible subgames

Definition

Let $(M \cup M', w_A)$ be an assignment game.
Let μ be an optimal matching, $I \subseteq M$ and $J \subseteq M'$.
The subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is μ-compatible if

$$w_A((M \setminus I) \cup (M' \setminus J)) + \sum_{i \in I \cap \mu^{-1}(M')} a_{i\mu(i)} + \sum_{j \in J \cap \mu(I)} a_{\mu^{-1}(j)j} = w_A(M \cup M').$$

✓ This implies that the restriction of μ to $(M \setminus I) \times (M' \setminus J)$ is optimal for the submarket.

✓In Example 3:

	1'	2'	3'
1	5	**8**	2
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$I = \{2\},\ J = \emptyset$
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The extended core of a compatible subgame

Let $(M \cup M', w_A)$ be an assignment game.

- For any fixed optimal matching μ, the principal section is

$$B^{\mu}(w_A) = \left\{ (u,v) \in I(w_A) \;\middle|\; \begin{array}{l} u_i + v_j = a_{ij} \text{ for all } (i,j) \in \mu \\ u_i = 0 \text{ if } i \text{ is unassigned by } \mu \\ v_j = 0 \text{ if } i \text{ is unassigned by } \mu \end{array} \right\}.$$

- If $((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}})$ is a μ-compatible subgame, its extended core is:

$$\hat{C}(w_{A_{-I\cup J}}) = \left\{ (u,v) \in \mathbb{R}^{M} \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} (u_{-I}, v_{-J}) \in C(w_{A_{-I\cup J}}), \\ u_i = a_{i\mu(i)} \text{ for all } i \in I \cap \mu^{-1}(M'), \\ v_j = a_{\mu^{-1}(j)j} \text{ for all } j \in J \cap \mu(I), \\ u_i = 0, v_j = 0 \text{ if unassigned by } \mu. \end{array} \right\}$$

- $\hat{C}(w_{A_{-I\cup J}}) \supseteq \left\{ (u,v) \in C(w_A) \;\middle|\; \begin{array}{l} u_i = a_{i\mu(i)} \text{ for all } i \in I, \\ v_j = a_{\mu^{-1}(j)j} \text{ for all } j \in J \end{array} \right\}$ and if $(M \cup M', w_A)$ has a dominant diagonal both sets coincide.

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Properties of the μ-compatible subgames

Fact

Let $(M \cup M', w_A)$ *be an assignment game and* μ *an optimal matching.*

1. $I \subseteq \{i \in M \mid \overline{u}_i^A = a_{i\mu(i)}\} \Rightarrow ((M \setminus I) \cup M', w_{A_{-I}})$ *μ-compat.*
2. *If* $i^* \in M$, *the subgame* $((M \setminus \{i^*\}) \cup M', w_{A_{-\{i^*\}}})$ *is μ-compatible if and only if* $\overline{u}_{i^*}^A = a_{i^*\mu(i^*)}$.

Corollary

Let $(M \cup M', w_A)$ *be an assignment game and* μ *an optimal matching.*

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Properties of the μ-compatible subgames

Let us define the set

$$\mathcal{C}_A^{\mu} = \{(I,J) \in 2^M \times 2^{M'} \mid ((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}}) \ \mu\text{-compatible}\}$$

Then,

Fact

If $(I_1, J_1) \in \mathcal{C}_A^{\mu}$ *and* $(I_2, J_2) \in \mathcal{C}_A^{\mu}$ *with* $I_1 \subseteq I_2$ *and* $J_1 \subseteq J_2$, *then*

$$(I_1, J_2) \in \mathcal{C}_A^{\mu} \text{ and } (I_2, J_1) \in \mathcal{C}_A^{\mu}.$$

The stable set in the μ-principal section

Theorem

Let $(M \cup M', w_A)$ and μ an optimal matching. The union of the extended cores of all the μ-compatible subgames, $V^{\mu}(w_A)$, is a stable set:

$$V^{\mu}(w_A) = \bigcup_{(I,J)\in \mathcal{C}_A^{\mu}} \hat{C}(w_{A_{-I\cup J}}),$$

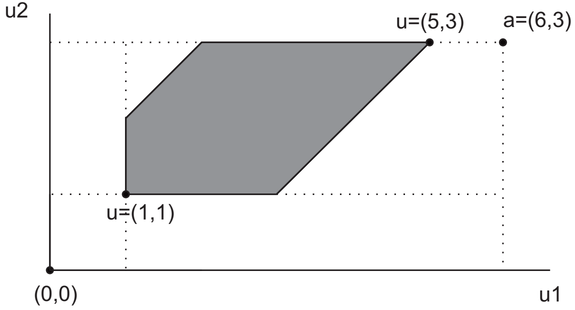
where
$\mathcal{C}_A^{\mu} = \{(I,J) \in 2^M \times 2^{M'} \mid ((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ *μ-compatible$\}$.*

$$(u,v) \in V^{\mu}(w_A) \Leftrightarrow \quad \text{for all } (i,j) \in \mu^{-1}(M') \times \mu(M) \text{ either}$$
$$u_i + v_j \geq a_{ij} \text{ or}$$
$$u_i = a_{i\mu(i)} \text{ or}$$
$$v_j = a_{\mu^{-1}(j)j}.$$

Example 2

	1'	2'	3'
1	**6**	2	1
2	4	**3**	1

✓ $(\overline{u}, \underline{v}) = (5, 3, 1, 0)$.
✓ $(\underline{u}, \overline{v}) = (1, 1, 5, 2)$.
✓ The core of this game is not stable.



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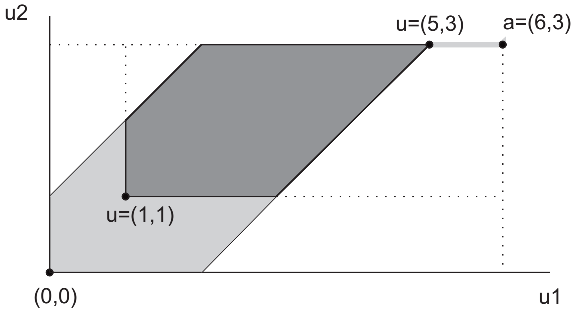

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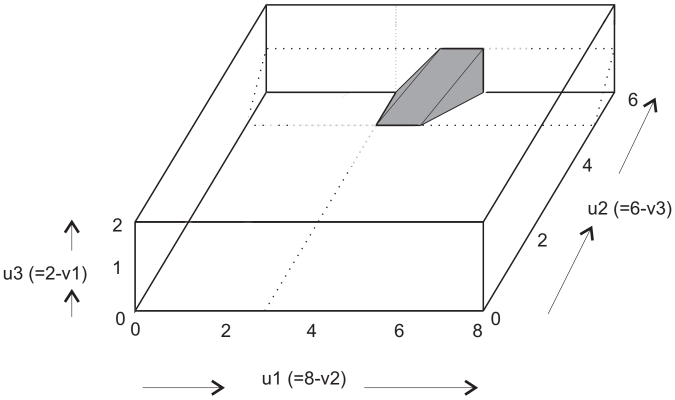

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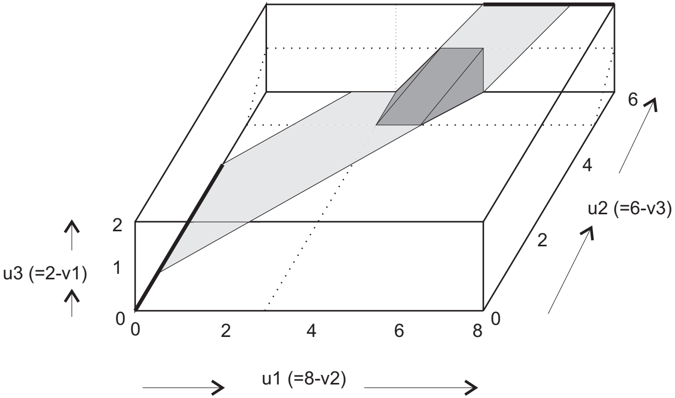

A sketch of the proof

✓ Shubik (1985) proves internal stability.

✓ To prove external stability we first define

$$R^{\mu}(w_A) = \{(u, v) \in B^{\mu}(w_A) \mid \underline{u}_i^A \leq u_i \leq \overline{u}_i^A, \forall i \in M\}$$

and notice that $C(w_A) \subseteq R^{\mu}(w_A) \subseteq B^{\mu}(w_A) \subseteq I(w_A)$.

Lemma (Shubik, 1985)

Let it be $(M \cup M', w_A)$ and μ an optimal matching

1. *There exists a piece-wise linear curve $L \subseteq V^{\mu}(w_A)$ through $(a, 0)$ and $(0, a)$.*
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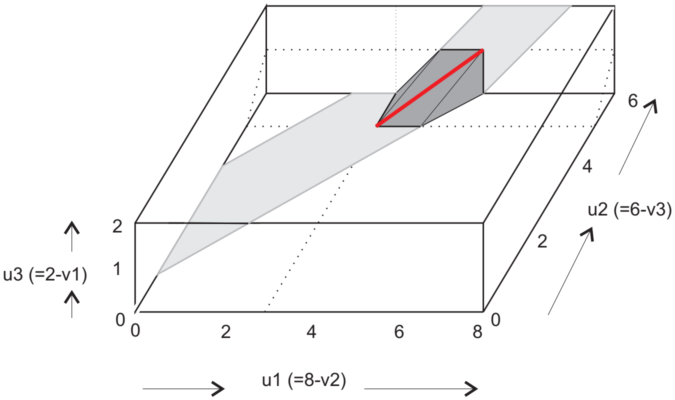

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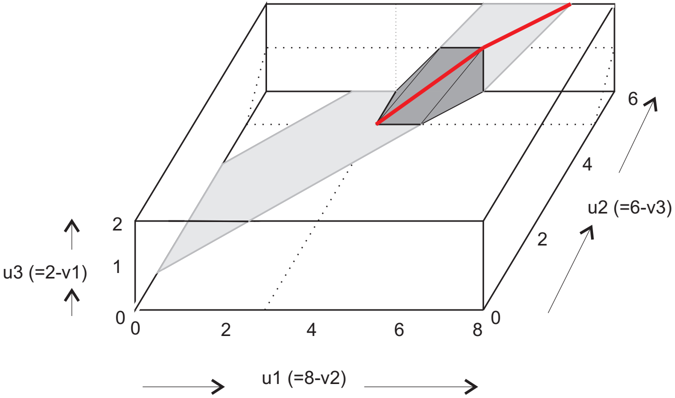

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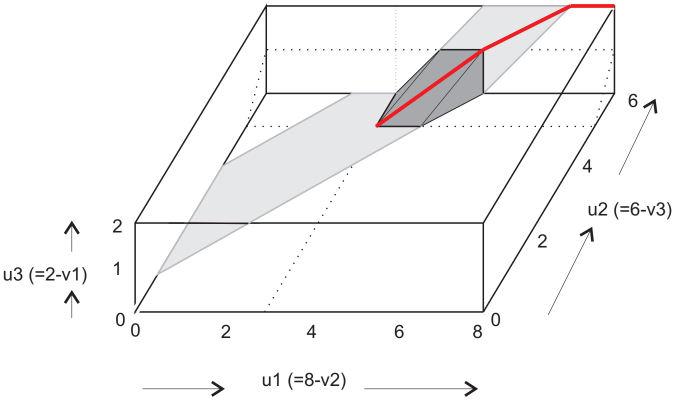

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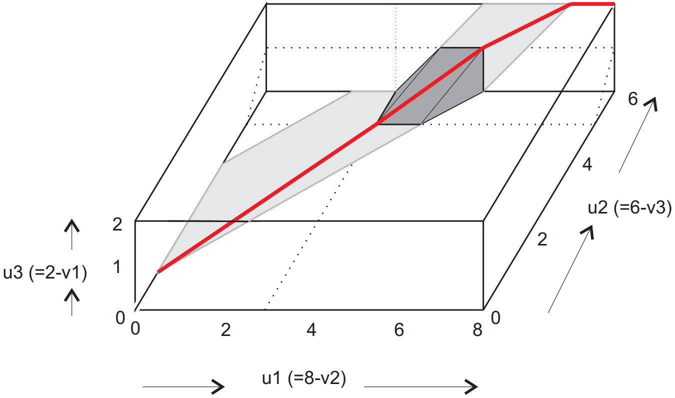

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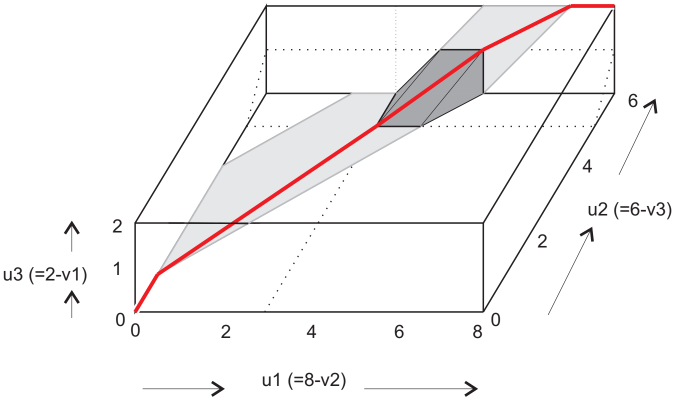

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Let it be $(M \cup M', w_A)$ *and* μ *an optimal matching*

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- From Núñez and Rafels (GEB, 2009), $C(w_A) = (\underline{u}, \underline{v}) + C(w_{A^e})$ where $(M \cup M', w_{A^e})$ has a dominant diagonal.
- Then, by Solymosi and Raghavan (IJGT, 2001), $(\tilde{u}, \tilde{v}) = (u, v) - (\underline{u}, \underline{v})$ is dominated by some $(x, y) \in C(w_{A^e})$ via $(p, q) \in M \times M'$ in the game w_{A^e}.
- We prove that (u, v) is also dominated by an element of $C(w_A)$ in the initial game.

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The last (and most difficult) part of the proof of the Theorem is how to dominate elements $(u, v) \in B^{\mu}(w_A) \setminus (V^{\mu}(w_A) \cup R^{\mu}(w_A))$.

- We construct a sequence of subsets of buyers

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_r \subseteq M$$

  such that, for all $0 \leq t \leq r$, $((M \setminus I_t) \cup M', w_{A_{-I_t}})$ is μ-compatible and (u_{-I_t}, v) belongs to $B^{\mu}(w_{A_{-I_t}})$.
- If $(u_{-I_r}, v) \in R^{\mu}(w_{A_{-I_r}})$, we apply the previous proposition.
- If $u_i > \overline{u}_i^{A_{-I_r}}$ for some $i \in M \setminus I_r$, then (u_{-I_r}, v) is dominated by the buyers-optimal core allocation of the μ-compatible subgame $((M \setminus I_r) \cup M'), w_{A_{-I_r}})$ or a "proximate point".
- If $u_i < \underline{u}_i^{A_{-I_r}}$ for some $i \in M \setminus I_r$, then (u_{-I_t}, v) is dominated by the sellers-optimal core allocation of some μ-compatible subgame $((M \setminus I_t) \cup M'), w_{A_{-I_t}})$, for $0 \leq t \leq r$ or a "proximate point".

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Final remarks

- The set $V^{\mu}(w_A)$ is the only stable set contained in the μ-principal section.
- In this stable set, third-party payments (with respect to μ) are excluded.

Theorem

$V^{\mu}(w_A)$ is also a lattice (with respect to the same partial order defined on the core).

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Final remarks

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