Noncooperative Game Theory
Extensive games

Julio González-Díaz

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Motivation

Strategic games may fail to capture dynamic aspects of games. Extensive games aim to extensively describe the (possibly) nonstatic situation. The game matrix is as follows:

- Stay Out, Stay Out: (0, 1)
- Stay Out, Enter: (1, 0)
- Enter, Stay Out: (0, 1)
- Enter, Enter: (-1, -1)

Sure? Done? Fight? Yield?
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![Strategic game representation:](image)
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Strategic game representation:

<table>
<thead>
<tr>
<th></th>
<th>SB</th>
<th>SY</th>
<th>DF</th>
<th>DY</th>
</tr>
</thead>
<tbody>
<tr>
<td>OO</td>
<td>0,1</td>
<td>0,1</td>
<td>0,1</td>
<td>0,1</td>
</tr>
<tr>
<td>OE</td>
<td>1,0</td>
<td>1,0</td>
<td>0,1</td>
<td>0,1</td>
</tr>
<tr>
<td>EO</td>
<td>−1,−1</td>
<td>1,0</td>
<td>−1,−1</td>
<td>1,0</td>
</tr>
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Noncooperative Game Theory: Extensive games

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(IMUS 2011)
The Setting

We restrict attention to finite games: length and width. Players are assumed to be rational. By rational player we mean one who:

1. knows what he wants
2. has the only objective of getting what he wants
3. is able to identify the strategies that best fit his objective
4. there is no bound in the complexity of the computations he can make or in the sophistication of his strategies

Players are expected utility maximizers.
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Outline

1. The Formal Model
2. Mixed Strategies vs. Behavior Strategies
   - Kuhn’s Theorem
3. Nash Equilibrium
   - Implications of Kuhn’s Theorem
4. Rethinking the Setting
   - Common Knowledge
   - Incomplete Information Games
5. Equilibrium Refinements
   - Perfect Information: Subgame Perfect Equilibrium
   - Imperfect Information: Sequential Equilibrium
## The Formal Model

1. **The Formal Model**

2. **Mixed Strategies vs. Behavior Strategies**
   - Kuhn’s Theorem

3. **Nash Equilibrium**
   - Implications of Kuhn’s Theorem

4. **Rethinking the Setting**
   - Common Knowledge
   - Incomplete Information Games

5. **Equilibrium Refinements**
   - Perfect Information: Subgame Perfect Equilibrium
   - Imperfect Information: Sequential Equilibrium
Extensive Games: Formal Model as in Selten (1975)
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Extensive Games

- Model
- Mixed vs. Behavior
- Nash Eq.
- Rethinking the Setting
- Refinements
- References

Noncooperative Game Theory: Extensive games

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Extensive Games: Formal Model as in Selten (1975)

Noncooperative Game Theory: Extensive games

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Extensive Games: Formal Model as in Selten (1975)

An extensive game is a 7-tuple \( \Gamma := (X, E, P, W, C, p, U) \). There are alternative representations, but these elements are implicitly or explicitly present in any extensive game.
Extensive Games: Formal Model as in Selten (1975)

A strategic game is a pair $G := (A, u)$
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Extensive Games: Formal Model as in Selten (1975)

\[ (X, E, P, W, C, p, U) \]

1. **Game tree**
2. **Player partition**
3. **Information partition**
4. **Choice partition**
5. **Probability assignment**
6. **Utilities**
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**Game tree.** \((X, E)\) is a finite tree:
- \(X\) is a finite set of nodes
- \(E \subset X \times X\) is a finite set of arcs (edges)
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- Root node
- Unique paths
- Terminal nodes \((Z)\)
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Player Partition.
\[ P := \{ P_0, P_1, \ldots, P_n \} \] is a partition of \( X \setminus Z \) that indicates, for each nonterminal node \( x \), which player has to make a decision at \( x \).
**Extensive Games: Formal Model** as in Selten (1975)

**Game tree**

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**Information partition**

**Choice partition**

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$\begin{align*}
x_0 &\xrightarrow{0.5} x_1 \xrightarrow{ND} x_2 \xrightarrow{ND} \{z^1, z^2, z^5, z^6\} \\
&\xrightarrow{0.5} x_2 \xrightarrow{D} x_3 \xrightarrow{D} \{z^3, z^4, z^7, z^8\} \\
&\xrightarrow{ND} x_1 \xrightarrow{ND} x_2 \xrightarrow{D} \{z^1, z^2, z^5, z^6\} \\
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Extensive Games: Formal Model as in Selten (1975)

$\begin{align*}
(x_0, E, P, W, C, p, U) \\
1 & \quad \text{Game tree} \\
2 & \quad \text{Player partition} \\
3 & \quad \textbf{Information partition} \\
4 & \quad \text{Choice partition} \\
5 & \quad \text{Probability assignment} \\
6 & \quad \text{Utilities}
\end{align*}$

$\begin{align*}
(z^1 & (-1, -1) \\
z^2 & (-15, 0) \\
z^3 & (0, -15) \\
z^4 & (-10, -10) \\
z^5 & (-1, -1) \\
z^6 & (0, -15) \\
z^7 & (-15, 0) \\
z^8 & (-10, -10)
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Extensive Games: Formal Model as in Selten (1975)

Information Partition. $W := \{W_1, \ldots, W_n\}$, each $W_i$ is a partition of $P_i$. Each information set $w \in W_i$ contains the nodes of $P_i$ in which player $i$ has the same information:
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- Same number of arcs starting from \( x \) and \( \hat{x} \) if \( x, \hat{x} \in w \)

\[ (X, E, P, W, C, p, U) \]

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Extensive Games: Formal Model as in Selten (1975)

Choice partition. $C$, a partition of the set of arcs starting outside $P_0$. Imposes that each player has to select an alternative at each decision node but he can only select a choice at each information set.
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2. Player partition
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The probability assignment. \(p\) is a map that assigns, to each \(x \in P_0\), a probability distribution \(p_x\), defined over the set of arcs starting at \(x\). Hence, \(p\) provides a description of nature moves.
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Utilities: $U := (U_1, \ldots, U_n)$, provides the utility functions of the players, defined over $Z$.
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Classes of Extensive Games
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Perfect information
An extensive game $\Gamma$ is a game with **perfect information** if, for each $i \in N$, each $w \in W_i$ contains exactly one node of $X \setminus Z$. 

Noncooperative Game Theory: Extensive games

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![Game Tree Diagram]

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Classes of Extensive Games

Perfect vs imperfect recall
Classes of Extensive Games
Perfect vs imperfect recall

A card game (Kuhn, 1953)

- Two teams:
  - \( T_1 \) → Players 1 and 3
  - \( T_2 \) → Player 2
Classes of Extensive Games
Perfect vs imperfect recall

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- After C, 3, uninformed, decides if 1 and 3 exchange cards

- Again, highest card wins 1$

**Imperfect Recall!**
Classes of Extensive Games

Perfect vs imperfect recall
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A card game (Kuhn, 1953)

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Classes of Extensive Games
Perfect vs imperfect recall

A card game (Kuhn, 1953)
Classes of Extensive Games

Perfect vs imperfect recall

A card game (Kuhn, 1953)

- Team: aligned interests, different players
Classes of Extensive Games

Perfect vs imperfect recall

A card game (Kuhn, 1953)

- Team: aligned interests, different players
- Imperfect recall, absent mindedness
Classes of Extensive Games

Perfect vs imperfect recall

A card game (Kuhn, 1953)

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Classes of Extensive Games
Perfect vs imperfect recall

A card game (Kuhn, 1953)

Perfect recall
An extensive game $\Gamma$ is a game with **perfect recall** if, for each $i \in N$ and each pair $w, \hat{w} \in W_i$:
If one node $x \in \hat{w}$ comes after a choice $c \in C_w$, then every node $\hat{x} \in \hat{w}$ comes after $c$
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**A card game (Kuhn, 1953)**

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$i$ remembers what he has known and done!!
Mixed Strategies vs. Behavior Strategies

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Strategies in Extensive Games

Let \( \Gamma = (X, E, P, W, C, p, U) \) be an extensive game.
Strategies in Extensive Games

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**Pure strategies**

A **pure strategy** $a_i$ of player $i$ is a map that assigns, to each information set $w \in W_i$, a choice $a_i(w) \in C_w$. 

**Behavior strategies**

A **behavior strategy** $b_i$ of player $i$ is a map that assigns, to each information set $w \in W_i$, a lottery over his choices in $C_w$. $b_i(c)$ is the probability that $i$ assigns to choice $c$ at $w$. $B_i$ is the set of behavior strategies of player $i$. $B_i \subset A_i$. $B := \prod_{i=1}^n B_i$ is the set of behavior strategy profiles $A \subset B$. 

Randomizing is important to get existence results
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$A_i \subset B_i$  \hspace{1cm} $A \subset B$
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Why behavior and not mixed?

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Noncooperative Game Theory: Extensive games

Julio González-Díaz (IMUS 2011) 18/48
Strategies in Extensive Games

Mixed strategies: Lotteries over pure strategies

Behavior strategies: Lotteries over choices

Utility of behavior strategy $b \in B$:
$$u_i(b) = \sum_{z \in Z} p(z, b) U_i(z)$$

Since $A \subset B$.

$\Gamma := (A, u)$ is a strategic game

Utility of mixed strategy $s \in S = \Delta A$:
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Extensive game $\Gamma$

Associated strategic game $G_{\Gamma}$

Mixed extension of the associated strategic game $E(G_{\Gamma})$

Equilibrium existence for $E(G_{\Gamma})$?

Noncooperative Game Theory: Extensive games
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Behavior vs. Mixed??
Equivalence of Strategies

The strategies $s_i \in S_i$ and $b_i \in B_i$ are:

- Equivalent: They induce same probabilities over $A_i$.
- Realization equivalent: They induce same probabilities over $X$ vs any strategy.
- Payoff equivalent: They induce same payoffs vs any strategy:

  For each $\hat{s}_{-i} \in S_{-i}$ and each $x \in X$,

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Every behavior strategy induces a lottery over pure strategies.
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Hence, for each behavior st, there is an **equivalent** mixed st
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s_2 = \frac{1}{2}(L_2, l_2) + \frac{1}{2}(R_2, r_2)
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**Correlation across inf. sets!**
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**Correlation across inf. sets!**

“$B_i \subset S_i$”
Equivalence of Strategies

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**Realization equivalent:** They induce same probabilities over $X$ →

**Payoff equivalent:** They induce same payoffs → vs any $\hat{s}_i \in S_i$

Every behavior strategy induces a lottery over pure strategies. The probability that $a_i$ is played when $i$ plays according to $b_i$ is

```
"probability of $a_i$" = \prod_{w \in W_i} b_i(a_i(w))
```

Hence, for each behavior st, there is an equivalent mixed st

```
s_2 = \frac{1}{2} (L_2, l_2) + \frac{1}{2} (R_2, r_2)
```

No equivalent $b_i \in B_i$

**Correlation across inf. sets!**

```
"B_i \subset S_i"
```
Kuhn’s Theorem

Let \( \Gamma \) be an extensive game with perfect recall. Let \( i \in N \) and let \( s_i \in S_i \). Then, there is \( b_i \in B_i \) such that \( s_i \) and \( b_i \) are realization equivalent.

This result is fundamental for the analysis of extensive games. Whatever a player can get with a mixed strategy can also be achieved by a behavior strategy. Equilibrium analysis is based on the notion of best response: All that matters is the probability of reaching each node. Under perfect recall, behavior strategies suffice for eq. analysis. Nice, since mixed strategies 'hide' the dynamic aspects of the game (defined on the 'static' strategic game).

Perfect recall + Behavior strategies
Kuhn’s Theorem

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Imperfect recall

Noncooperative Game Theory: Extensive games
Kuhn’s Theorem

The role of perfect recall

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Imperfect recall

Perfect recall

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**Imperfect recall**

$$s_2 = \frac{1}{2}(L_1, L_2) + \frac{1}{2}(R_1, R_2)$$

**Perfect recall**

$$b_2$$

Noncooperative Game Theory: Extensive games

Julio González-Díaz (IMUS 2011) 23/48
Kuhn’s Theorem

The role of perfect recall

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Imperfect recall

Perfect recall

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- $b_2$?
Kuhn’s Theorem
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Kuhn’s Theorem

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Noncooperative Game Theory: Extensive games
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**Perfect recall**

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Kuhn’s Theorem
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**Imperfect recall**

![Imperfect recall diagram]

- $s_2 = \frac{1}{2}(L_1, L_2) + \frac{1}{2}(R_1, R_2)$
- $b_2$?

**Perfect recall**

![Perfect recall diagram]

- $s_2 = \frac{1}{2}(L_1, L_2, l_2) + \frac{1}{2}(R_1, R_2, r_2)$
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**Perfect recall**

- $s_2 = \frac{1}{2}(L_1, L_2, l_2) + \frac{1}{2}(R_1, R_2, r_2)$
- $b_2$?
Kuhn’s Theorem

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Let $\Gamma$ be an extensive game with perfect recall. Let $i \in N$ and let $s_i \in S_i$. Then, there is $b_i \in B_i$ such that $s_i$ and $b_i$ are realization equivalent.

\begin{itemize}
  \item $s_2 = \frac{1}{2}(L_1, L_2) + \frac{1}{2}(R_1, R_2)$
  \item $b_2$
\end{itemize}

Noncooperative Game Theory: Extensive games

Julio González-Díaz  (IMUS 2011) 23/48
Kuhn’s Theorem

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Let $\Gamma$ be an extensive game with **perfect recall**. Let $i \in N$ and let $s_i \in S_i$. Then, there is $b_i \in B_i$ such that $s_i$ and $b_i$ are realization equivalent.

### Imperfect recall

- $s_2 = \frac{1}{2}(L_1, L_2) + \frac{1}{2}(R_1, R_2)$
- $b_2$

### Perfect recall

- $s_2 = \frac{1}{2}(L_1, L_2, l_2) + \frac{1}{2}(R_1, R_2, r_2)$
- $b_2$
Nash Equilibrium

1. The Formal Model
2. Mixed Strategies vs. Behavior Strategies
   - Kuhn’s Theorem
3. Nash Equilibrium
   - Implications of Kuhn’s Theorem
4. Rethinking the Setting
   - Common Knowledge
   - Incomplete Information Games
5. Equilibrium Refinements
   - Perfect Information: Subgame Perfect Equilibrium
   - Imperfect Information: Sequential Equilibrium
Nash equilibrium

A behavior strategy $b^*$ $\in B$ is a Nash equilibrium of $\Gamma$ if, for each $i \in N$ and each $\hat{b}_i \in B_i$,

$$u_i(b^*) \geq u_i(b^* - i, \hat{b}_i)$$

Lemma

A pure strategy $a \in A$ is a Nash equilibrium of $\Gamma$ if and only if it is a Nash equilibrium of $G_{\Gamma}$

Theorem

Let $\Gamma$ be an extensive game with perfect recall. Then, $\Gamma$ has, at least, one Nash equilibrium (in behavior strategies)

Proof.

By Nash's Theorem, $E(G_{\Gamma})$ has a Nash equilibrium in mixed strategies, $s^* \in S$.

By Kuhn's Theorem, there is $b^* \in B$ realization equivalent to $s^*$.

Then, $s^*$ and $b^*$ are also payoff equivalent and $b^*$ is a Nash equilibrium of $\Gamma$.
Nash equilibrium

A behavior strategy \( b^* \in B \) is a **Nash equilibrium** of \( \Gamma \) if, for each \( i \in N \) and each \( \hat{b}_i \in B_i \),

\[
u_i(b^*) \geq u_i(b^*_{-i}, \hat{b}_i)\]
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A behavior strategy \( b^* \in B \) is a **Nash equilibrium** of \( \Gamma \) if, for each \( i \in \mathcal{N} \) and each \( \hat{b}_i \in B_i \),

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- By **Nash’s Theorem**, $E(G_{\Gamma})$ has a Nash equilibrium in mixed strategies, $s^* \in S$
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A behavior strategy \( b^* \in B \) is a **Nash equilibrium** of \( \Gamma \) if, for each \( i \in N \) and each \( \hat{b}_i \in B_i \),

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**Proof.**

- By **Nash’s Theorem**, \( E(G_\Gamma) \) has a Nash equilibrium in mixed strategies, \( s^* \in S \)
- By **Kuhn’s Theorem**, there is \( b^* \in B \) realization equivalent to \( s^* \)
Nash equilibrium

A behavior strategy \( b^* \in B \) is a **Nash equilibrium** of \( \Gamma \) if, for each \( i \in N \) and each \( \hat{b}_i \in B_i \),

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- Then, \( s^* \) and \( b^* \) are also payoff equivalent and \( b^* \) is a Nash equilibrium of \( \Gamma \)
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If $b^* \in B$ is a Nash equilibrium of an extensive game $\Gamma$ with perfect recall, then it is a Nash equilibrium of $E(G_\Gamma)$
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**Proof.** Suppose not.
Nash equilibrium

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- Let \( i \in N \) and \( s_i \in S_i \) be such that \( u_i(b^*) < u_i(b^*_{-i}, s_i) \)
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**Proof.** Suppose not.

- Let $i \in N$ and $s_i \in S_i$ be such that $u_i(b^*) < u_i(b_{-i}^*, s_i)$
- By Kuhn’s Theorem, there is $b_i \in B$ realization equivalent to $s_i$
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**Proof.** Suppose not.

- Let \( i \in N \) and \( s_i \in S_i \) be such that \( u_i(b^*) < u_i(b^*_{-i}, s_i) \)
- By **Kuhn’s Theorem**, there is \( b_i \in B \) realization equivalent to \( s_i \)
- Since \( b_i \) and \( s_i \) are payoff equivalent, \( u_i(b^*) < u_i(b^*_{-i}, b_i) \)
$b^* \in B$ is a **Nash equilibrium** of $\Gamma$ if, for each $i \in N$ and each $\hat{b}_i \in B_i$,

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---

A card game (Kuhn, 1953)

<table>
<thead>
<tr>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>$T_1$</td>
<td>$H$</td>
</tr>
</tbody>
</table>

Kind of "matching pennies"

Unique mixed Nash equilibrium: $s_1 = (0.5, 0.5, 0)$, $s_2 = (0, 0.5, 0.5)$

No realization equivalent behavior strategy
\( b^* \in B \) is a **Nash equilibrium** of \( \Gamma \) if, for each \( i \in N \) and each \( \hat{b}_i \in B_i \),

\[
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\]

**Theorem**

Let \( \Gamma \) be an extensive game with **perfect recall**. Then, \( \Gamma \) has, at least, one Nash equilibrium (in behavior strategies)

**Theorem**

If \( b^* \in B \) is a Nash equilibrium of an extensive game \( \Gamma \) with **perfect recall**, then it is a Nash equilibrium of \( E(G_{\Gamma}) \)
$b^* \in B$ is a **Nash equilibrium** of $\Gamma$ if, for each $i \in N$ and each $\hat{b}_i \in B_i$,

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**Theorem**

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---

**A card game (Kuhn, 1953)**

```
T1

T2

0.5

0.5

S

E

w_1^1

w_2^1

w_1^2

w_2^2

C

NE

NE

S

C

(1, -1)

(2, -2)

(0, 0)

(-2, 2)

(0, 0)

(-1, 1)
```

Kind of “matching pennies”

Unique mixed Nash equilibrium: $s_1 = (0, 0.5, 0.5, 0)$, $s_2 = (0.5, 0, 0.5, 0)$

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$b^* \in B$ is a **Nash equilibrium** of $\Gamma$ if, for each $i \in N$ and each $\hat{b}_i \in B_i$,

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<tbody>
<tr>
<td>$SNE$</td>
<td>0, 0</td>
<td>$-0.5$, 0.5</td>
</tr>
<tr>
<td>$SE$</td>
<td>0, 0</td>
<td>0.5, $-0.5$</td>
</tr>
<tr>
<td>$CNE$</td>
<td>0.5, $-0.5$</td>
<td>0, 0</td>
</tr>
<tr>
<td>$CE$</td>
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<td>0, 0</td>
</tr>
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---

Kind of “matching pennies”

Unique mixed Nash equilibrium: $s_1 = (0, 0.5, 0.5, 0)$ and $s_2 = (0.5, 0, 0.5, 0)$

No realization equivalent behavior strategy
$b^* \in B$ is a **Nash equilibrium** of $\Gamma$ if, for each $i \in N$ and each $\hat{b}_i \in B_i$,

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**Theorem**

Let $\Gamma$ be an extensive game with perfect recall. Then, $\Gamma$ has, at least, one Nash equilibrium (in behavior strategies).

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If $b^* \in B$ is a Nash equilibrium of an extensive game $\Gamma$ with perfect recall, then it is a Nash equilibrium of $E(G_\Gamma)$

**A card game (Kuhn, 1953)**

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![Card game diagram](image)
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Kind of “matching pennies”

Unique mixed Nash equilibrium:
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Equivalent representations

Most equilibrium concepts are sensible to the chosen representation for the game.
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Explicit assumptions

- Finiteness of the game tree
- Players' rationality
- Players are expected utility maximizers

Implicit assumptions

- Common knowledge of rationality
- Common knowledge of all the elements of the model

Utilities, Timing of the game, Actions of the players, Set of players...
Rethinking the setting

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- Common knowledge + expected utility ⇒ ordinal vs cardinal
Degrees of Knowledge and Common Knowledge

Coordinated attack problem (Halpern, 1986; Rubinstein, 1989)

Two allied armies are on opposite hills waiting to attack their enemy. Commander in chief in one, a captain at the other. In order to ensure a successful battle, none of them will attack unless he is sure that the other will do so at the same time. The commander sends a messenger to the captain with the message "I plan to attack at night." Messenger informs the captain and gets captured on the way back. Both the commander and the captain know that "the commander plans to attack at night." But the commander does not know that the captain knows it. The captain does not know if the commander know that he knows it. Can they be certain that both will attack? Should they attack?

If the event "the commander plans to attack at night" were common knowledge, there would be no doubt.
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Degrees of Knowledge and Common Knowledge

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There is a lot of research studying the (strategic) implications of (weakenings of) common knowledge.

- Common knowledge of rationality
- Common knowledge of other events
- Common prior

The assumption of common knowledge IS NOT innocuous.
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Games with Incomplete Information
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Noncooperative Game Theory: Extensive games

Julio González-Díaz

(IMUS 2011)
Games with Incomplete Information

Noncooperative Game Theory: Extensive games

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Noncooperative Game Theory: Extensive games

Julio González-Díaz

(IMUS 2011)
Every form of Incomplete information can be reduced to imperfect information (Harsanyi, 1967-68)
Games with Incomplete Information

- **Every form of** Incomplete information can be reduced to imperfect information (Harsanyi, 1967-68)

Utilities
- (1, 1)
- (−50, 0)
- (0, −50)
- (0, 0)
- (1, −1)
- (−50, 0)
- (0, 50)
- (0, 0)

Players
- Normal
- Crazy

Actions
- Nice
- Rough

Game tree
- Bayesian games
- Possibly infinite games

Noncooperative Game Theory: Extensive games

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Utilities
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Beliefs
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### Bayesian games

- Utilities
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Bayesian games

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- **Utilities**
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**Games with Incomplete Information**

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![Game Tree](attachment:image.png)

- **Utilities**
- **Players**
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Bayesian games

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Nobel Prize Economics (1994): Nash, Selten, Harsanyi
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Weakness of Nash equilibrium in extensive games
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(Imus 2011) 36/48
Equilibrium Refinements
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Noncooperative Game Theory: Extensive games
Julio González-Díaz (IMUS 2011) 36/48
Equilibrium Refinements

Weakness of Nash equilibrium in extensive games

\begin{itemize}
\item[(0, 1)]
\item[(−1, −1)]
\item[(1, 0)]
\end{itemize}
Equilibrium Refinements
Weakness of Nash equilibrium in extensive games

The chain store game (Selten 1978)

\[
\begin{align*}
\text{Node 2 is not on the path of play.}
\end{align*}
\]

Behavior at node 2 is irrational.

Incredible threats...
Equilibrium Refinements
Weakness of Nash equilibrium in extensive games

The chain store game (Selten 1978)

Node 2 is not on the path of play. Behavior at node 2 is irrational. In a repeated setting, reputation of being tough, etc.
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**The chain store game (Selten 1978)**

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\( (O, F) \) is a Nash equilibrium
Equilibrium Refinements

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The chain store game (Selten 1978)

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\[(O, F)\] is a Nash equilibrium

Node 2 is not on the path of play

Behavior at node 2 is irrational. Incredible threats
Equilibrium Refinements
Weakness of Nash equilibrium in extensive games

The chain store game (Selten 1978)

(0, 1)

stay Out

(1, 0)

Fight

Node 2 is not on the path of play

Yield

Behavior at node 2 is irrational. Incredible threats

(0, F) is a Nash equilibrium

Not so in a repeated setting, reputation of being tough, . . .
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The chain store game (Selten 1978)

(\(O, F\)) is a Nash equilibrium

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Several approaches in the spirit of the refinements for strategic games: strict, perfect, proper, ... (Selten 1975, van Damme 1984)

Using the dynamic structure of the game through backward reasoning, backward induction: subgame perfect equilibrium, sequential equilibrium (Selten 1975, Kreps and Wilson 1982)

Using the dynamic structure of the game through forward reasoning, forward induction: stable sets (Kohlberg and Mertens)

Combinations of the above approaches, other approaches, ...
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Perfect and Proper Equilibrium in Extensive Games
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Perfect Equilibrium

- Perfect in $\Gamma$ (Trembles defined directly on the game tree)
- Perfect in $E(G_\Gamma)$
Perfect and Proper Equilibrium in Extensive Games

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- Perfect in $\Gamma$ (Trembles defined directly on the game tree)
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- Perfect in $E(G_{A\Gamma})$
Perfect and Proper Equilibrium in Extensive Games

Perfect Equilibrium
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Every extensive game with perfect recall has a perfect equilibrium
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Corollary
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Perfect in $\Gamma$ $\iff$ Perfect in $E(G_{A\Gamma})$

Corollary
Every extensive game with perfect recall has a perfect equilibrium

Proper Equilibrium
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Perfect and Proper Equilibrium in Extensive Games

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- Perfect in $\Gamma$ (Trembles defined directly on the game tree)
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Perfect in $\Gamma \iff$ Perfect in $E(G_{AG})$

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Every extensive game with perfect recall has a perfect equilibrium

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Similar ideas to those in strategic games
Perfect and Proper Equilibrium in Extensive Games

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Similar ideas to those in strategic games
Capture very well dynamic aspects
Perfect and Proper Equilibrium in Extensive Games

**Perfect Equilibrium**

Originally defined for extensive games!

- Perfect in $\Gamma$ (Trembles defined directly on the game tree)
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- Perfect in $E(G_{AG})$

**Proposition**

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Similar ideas to those in strategic games

Capture very well dynamic aspects
Backward and Forward Reasoning

**Backward Induction**

1. **Stay Out**
   - (0, 1)

2. **Fight**
   - (−1, −1)

**Forward Induction**

1. **Enter**
   - (1, 0)

2. **Yield**
   - (1, 0)

**Idea of subgame perfection**

- **L1**
- **R1**

- **L2**
- **R2**

- (3, 3)
- (4, 1)
- (1, 0)
- (0, 0)
- (2, 1)
Backward and Forward Reasoning

**Backward Induction**

1. 1: stay Out (0, 1)
2. 2: Fight (−1, −1)
3. 2: Yield (1, 0)

**Forward Induction**

1. 1
   - C1: (0, 0)
   - L1: (1, 0)
2. 2
   - R1: (2, 1)
   - L2: (0, 0)
   - R2: (1, 0)

Idea of subgame perfection

Hard to refine
Backward and Forward Reasoning

**Backward Induction**

- **Fight**
  - 2
  - (−1, −1)
- **Yield**
  - 2
  - (1, 0)

**Forward Induction**

- 1
  - L₁
  - (0, 0)
  - R₁
  - (2, 1)
- 2
  - L₂
  - (4, 1)
  - R₂
  - (1, 0)
  - C₁
  - (3, 3)
Backward and Forward Reasoning

**Backward Induction**

- **Node 2**:
  - Action: Fight
  - Payoff: \((-1, -1)\)
- **Node 2**:
  - Action: Yield
  - Payoff: \((1, 0)\)

**Forward Induction**

- **Node 1**:
  - Action: Left
  - Payoff: \((4, 1)\)
- **Node 1**:
  - Action: Right
  - Payoff: \((1, 0)\)
- **Node 2**:
  - Action: Left
  - Payoff: \((0, 0)\)
- **Node 2**:
  - Action: Right
  - Payoff: \((2, 1)\)
- **Node 2**:
  - Action: Right
  - Payoff: \((3, 3)\)
Backward and Forward Reasoning

**Backward Induction**

1. Stay Out → (0, 1)
2. Enter → (1, 0)

**Forward Induction**

1. C1
   - L1 2 → (0, 0)
   - R1 2 → (2, 1)
2. L2 → (4, 1)
   - R2 → (1, 0)

Idea of subgame perfection

Hard to refine
Backward and Forward Reasoning

**Backward Induction**

Stay Out → (0, 1)

Enter → (1, 0)

**Forward Induction**

1

- \(L_2\) → (4, 1)
- \(R_2\) → (1, 0)

2

- \(L_1\) → (0, 0)
- \(R_1\) → (2, 1)

(1, 0)

(2, 1)

(3, 3)
Backward and Forward Reasoning

Backward Induction

Forward Induction

<table>
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<td>Fight</td>
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</tr>
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</tr>
<tr>
<td>(3, 3)</td>
<td></td>
</tr>
</tbody>
</table>
Backward and Forward Reasoning

- **Backward Induction**

1. **Stay Out** → (0, 1)
2. **Fight** → (−1, −1)
3. **Yield** → (1, 0)

- **Forward Induction**

1. **L1** → (1, 0)
2. **R2** → (0, 0)
3. **L2** → (4, 1)
4. **R2** → (2, 1)
5. **(3, 3)**

- Idea of subgame perfection

Noncooperative Game Theory: Extensive games
Backward and Forward Reasoning

**Backward Induction**

1. Enter: (1, 0)
2. Stay Out: (0, 1)
   - Fight: (-1, -1)
   - Yield: (1, 0)

**Forward Induction**

1. Enter: (1, 0)
2. Stay Out: (0, 0)
   - Fight: (4, 1)
   - Yield: (2, 1)

- Idea of subgame perfection

Noncooperative Game Theory: Extensive games

Julio González-Díaz (IMUS 2011)
Backward and Forward Reasoning

**Backward Induction**

- Idea of subgame perfection

**Forward Induction**

- Hard to refine

Stay Out

Fight

Yield

\[(0, 1)\]

\[(-1, -1)\]

\[(1, 0)\]

\[(3, 3)\]

\[(3, 3)\]
Subgame Perfect Equilibrium
Subgame Perfect Equilibrium

\[ F(x) \] is the set of nodes that come after \( x \) in the game tree.
Subgame Perfect Equilibrium

\( F(x) \) is the set of nodes that come after \( x \) in the game tree (a node comes after itself)
Subgame Perfect Equilibrium

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The game \( \Gamma \) can be decomposed at \( x \) if there is no information set simultaneously containing nodes of \( F(x) \) and nodes of \( X \setminus F(x) \).
Subgame Perfect Equilibrium

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\[
\begin{array}{c}
L_1 \quad r_1 \\
L_2 \quad R_2 \\
(3, 3) \quad (1, 0) \quad (0, 0) \quad (2, 1)
\end{array}
\]
Subgame Perfect Equilibrium

Γ is a subgame of Γ, let b ∈ B. Then, b is the strategy profile in Γ induced by b.

A subgame perfect equilibrium of Γ is a strategy profile b ∈ B such that, for each subgame Γ of Γ, b is a Nash equilibrium of Γ.
Subgame Perfect Equilibrium

- $\Gamma_x$ the game that $\Gamma$ induces in the tree whose root node is $x$
Subgame Perfect Equilibrium

- $\Gamma_x$ the game that $\Gamma$ induces in the tree whose root node is $x$
- $\Gamma_x$ is a subgame of $\Gamma$
Subgame Perfect Equilibrium

- $\Gamma_x$ the game that $\Gamma$ induces in the tree whose root node is $x$
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- Let $b \in B$. Then, $b_x$ is the strategy profile in $\Gamma_x$ induced by $b$
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A subgame perfect equilibrium of $\Gamma$ is a strategy profile $b \in B$ such that, for each subgame $\Gamma_x$ of $\Gamma$, $b_x$ is a Nash equilibrium of $\Gamma_x$. 
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**Lemma**
Every subgame perfect equilibrium is a Nash equilibrium

**Proof.** A game is a subgame of itself

**Proposition**
Every perfect equilibrium of $\Gamma$ is subgame perfect

**Idea.** Trembles 'put' all subgames on the path

**Corollary**
Every extensive game with perfect recall has a subgame perfect equilibrium

**Proposition**
Every extensive game with perfect information has a subgame perfect equilibrium in pure strategies

**Idea.** Backward induction
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**Corollary**
Every extensive game with *perfect recall* has a subgame perfect equilibrium
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**Idea.** Backward induction
Weakness of Subgame Perfect Equilibrium

Imperfect information games

\[
\begin{array}{ccc}
\text{L}_1 & \text{C}_1 & \text{R}_1 \\
\text{L}_2 & \text{R}_2 & \text{L}_2 \\
(3, 1) & (0, 0) & (0, 2) \\
(1, 1) & (2, 3) & \\
(\text{R}_1, \text{R}_2) & & \\
\end{array}
\]

is a Nash equilibrium, but \(\text{R}_2\) is a strictly dominated choice.

No subgames. Nash equilibrium \(\iff\) Subgame perfect equilibrium

\((\text{R}_1, \text{R}_2)\) is a subgame perfect equilibrium.

Backward induction arguments \(\rightarrow (\text{L}_1, \text{L}_2)\)

Noncooperative Game Theory: Extensive games

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Weakness of Subgame Perfect Equilibrium

Imperfect information games

\[ (1, 1) \]

\[ (2, 3) \]

\[ (0, 2) \]

\[ (0, 0) \]

\[ (3, 1) \]

\[ R_2 \]

\[ L_2 \]

\[ L_1 \]

\[ R_1 \]

\[ C_1 \]
Weakness of Subgame Perfect Equilibrium

Imperfect information games

```
(3, 1)
(0, 0)
(0, 2)
(1, 1)
(2, 3)
```

(R_1, R_2) is a Nash equilibrium, but R_2 is a strictly dominated choice.

No subgames. Nash equilibrium $\iff$ Subgame perfect equilibrium

Backward induction arguments $\rightarrow$ (L_1, L_2)
Weakness of Subgame Perfect Equilibrium

Imperfect information games

\[
\begin{array}{c}
L_1 & \rightarrow & R_1 \\
& & (2, 3) \\
L_2 & \rightarrow & R_2 \\
& & (1, 1) \\
& \rightarrow & L_2 \\
& & (0, 2) \\
C_1 & \rightarrow & R_2 \\
& & (0, 0) \\
& \rightarrow & L_1 \\
& & (3, 1)
\end{array}
\]

\( (R_1, R_2) \) is a Nash equilibrium
Weakness of Subgame Perfect Equilibrium

Imperfect information games

- $(R_1, R_2)$ is a Nash equilibrium, but $R_2$ is a strictly dominated choice
Weakness of Subgame Perfect Equilibrium

Imperfect information games

(R₁, R₂) is a Nash equilibrium, but R₂ is a strictly dominated choice

No subgames. Nash equilibrium ⇐⇒ Subgame perfect equilibrium
**Weakness of Subgame Perfect Equilibrium**

Imperfect information games

- $(R_1, R_2)$ is a Nash equilibrium, but $R_2$ is a **strictly dominated** choice
- No subgames. Nash equilibrium $\iff$ Subgame perfect equilibrium
- $(R_1, R_2)$ is a subgame perfect equilibrium
Weakness of Subgame Perfect Equilibrium

Imperfect information games

(R₁, R₂) is a Nash equilibrium, but R₂ is a **strictly dominated** choice

No subgames. Nash equilibrium \(\iff\) Subgame perfect equilibrium

(R₁, R₂) is a subgame perfect equilibrium

Backward induction arguments
Weakness of Subgame Perfect Equilibrium

Imperfect information games

- \((R_1, R_2)\) is a Nash equilibrium, but \(R_2\) is a **strictly dominated** choice
- No subgames. Nash equilibrium \(\iff\) Subgame perfect equilibrium
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A subgame perfect equilibrium of $\Gamma$ is a strategy profile $b \in B$ such that, for each subgame $\Gamma_x$ of $\Gamma$, $b_x$ is a Nash equilibrium of $\Gamma_x$. 
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Nash equilibrium

Every player maximizes his (expected) payoff when taking the strategies of the opponents as given
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- Not enough in a dynamic setting
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Nash equilibrium

Every player maximizes his (expected) payoff when taking the strategies of the opponents as given

- Not enough in a dynamic setting

Subgame perfect equilibrium

Every player maximizes his (expected) payoff, at every subgame, when taking the strategies of the opponents as given
Subgame Perfect Equilibrium

A **subgame perfect equilibrium** of \( \Gamma \) is a strategy profile \( b \in B \) such that, for each subgame \( \Gamma_x \) of \( \Gamma \), \( b_x \) is a Nash equilibrium of \( \Gamma_x \).

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Every player maximizes his (expected) payoff when taking the strategies of the opponents as given

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Subgame perfect equilibrium

Every player maximizes his (expected) payoff, **at every subgame**, when taking the strategies of the opponents as given

- Not enough in a setting with imperfect information
Subgame Perfect Equilibrium

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- Not enough in a dynamic setting

**Subgame perfect equilibrium**

Every player maximizes his (expected) payoff, **at every subgame**, when taking the strategies of the opponents as given
- Not enough in a setting with imperfect information

**Sequential equilibrium** (*Kreps and Wilson 1982*)

Every player maximizes his (expected) payoff, **at every information set**, when taking the strategies of the opponents as given
Subgame Perfect Equilibrium

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Nash equilibrium

Every player maximizes his (expected) payoff when taking the strategies of the opponents as given

- Not enough in a dynamic setting

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Every player maximizes his (expected) payoff, **at every subgame**, when taking the strategies of the opponents as given

- Not enough in a setting with **imperfect information**

Sequential equilibrium (Kreps and Wilson 1982)

Every player maximizes his (expected) payoff, **at every information set**, when taking the strategies of the opponents as given

- ... but, given what **beliefs**?
Weak Perfect Bayesian Equilibrium

Sequential equilibrium (Kreps and Wilson 1982)
Every player maximizes his (expected) payoff, at every information set, when taking the strategies of the opponents as given
Weak Perfect Bayesian Equilibrium

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Every player maximizes his (expected) payoff, at every information set, when taking the strategies of the opponents as given

\[ \text{...but, given what beliefs?} \]

\[ \begin{array}{l}
L_2 & (3, 1) \\
R_2 & (0, 0) \\
\end{array} \]

\[ \begin{array}{l}
L_2 & (0, 2) \\
R_2 & (1, 1) \\
\end{array} \]

Noncooperative Game Theory: Extensive games

Julio González-Díaz (IMUS 2011) 45/48
Weak Perfect Bayesian Equilibrium

Sequential equilibrium (Kreps and Wilson 1982)

Every player maximizes his (expected) payoff, at every information set, when taking the strategies of the opponents as given

- ...but, given what beliefs?

![Game tree diagram](image-url)

- $L_1$ and $L_2$
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![Game tree diagram]

- $L_1$ and $L_2 \rightarrow (1, 0)$
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\[
\begin{array}{c|c|c}
\text{Player 1} & \text{Player 2} & \text{Payoff} \\
\hline
L_1 & (3, 1) & \\
C_1 & (0, 0) & \\
R_1 & (0, 2) & \\
\hline
L_2 & (1, 1) & \\
R_2 & (1, 0) & \\
\end{array}
\]

• \(L_1\) and \(L_2\) \(\rightarrow\) \((1, 0)\)

• \(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\) and \(L_2\)
Weak Perfect Bayesian Equilibrium

Sequential equilibrium (Kreps and Wilson 1982)
Every player maximizes his (expected) payoff, at every information set, when taking the strategies of the opponents as given

- ...but, given what beliefs?

\[
\begin{align*}
(2, 3) & \quad (1, 1) \\
(0, 2) & \quad (0, 0) \\
(3, 1) & \\
& \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \\
& \quad L_1 \quad R_2 \quad L_2
\end{align*}
\]

- \( L_1 \) and \( L_2 \) \( \rightarrow \) \( (1, 0) \)
- \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) and \( L_2 \)
  - Bayes rule \( \iff \) Bayesian updating
Weak Perfect Bayesian Equilibrium

Sequential equilibrium (Kreps and Wilson 1982)

Every player maximizes his (expected) payoff, at every information set, when taking the strategies of the opponents as given

- ...but, given what beliefs?

- $L_1$ and $L_2 \rightarrow (1, 0)$
- $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $L_2 \rightarrow (0.5, 0.5)$
- Bayes rule $\iff$ Bayesian updating
Weak Perfect Bayesian Equilibrium

Sequential equilibrium (Kreps and Wilson 1982)

Every player maximizes his (expected) payoff, at every information set, when taking the strategies of the opponents as given

- ... but, given what beliefs?

\[
\begin{align*}
(1, 0) & \quad \text{if } L_1 \text{ and } L_2 \\
(0.5, 0.5) & \quad \text{if } (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \text{ and } L_2 \\
(1, 1) & \quad \text{if } R_1 \text{ and } L_2
\end{align*}
\]

Bayes rule $\iff$ Bayesian updating
Weak Perfect Bayesian Equilibrium

Sequential equilibrium (Kreps and Wilson 1982)

Every player maximizes his (expected) payoff, at every information set, when taking the strategies of the opponents as given

• ...but, given what beliefs?

\begin{itemize}
\item \( L_1 \) and \( L_2 \) \(\rightarrow\) (1, 0)
\item \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) and \( L_2 \) \(\rightarrow\) (0.5, 0.5)
\item Bayes rule \(\iff\) Bayesian updating
\item \( R_1 \) and \( L_2 \) \(\rightarrow\) (?, ?)
\end{itemize}
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- An assessment \((b, \mu)\) is sequentially rational if, for each \(i \in N\) and each \(w \in W_i\), \(b_i\) is a best reply of player \(i\) against \((b, \mu)\) at \(w\).
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- An assessment \((b, \mu)\) is weakly consistent with Bayes rule if \(\mu\) is derived using Bayesian updating in the path of \(b\)

Weak perfect Bayesian equilibrium

A weak perfect Bayesian equilibrium is an assessment that is sequentially rational and weakly consistent with Bayes rule
Sequential Equilibrium (Kreps and Wilson 1982)

- An **assessment** is a pair \((b, \mu)\), where \(b\) is a behavior strategy profile and \(\mu\) is a system of beliefs.

- An assessment \((b, \mu)\) is **sequentially rational** if, for each \(i \in N\) and each \(w \in W_i\), \(b_i\) is a best reply of player \(i\) against \((b, \mu)\) at \(w\).

- An assessment \((b, \mu)\) is **weakly consistent with Bayes rule** if \(\mu\) is derived using Bayesian updating in the path of \(b\).
Sequential Equilibrium (Kreps and Wilson 1982)

- An assessment is a pair \((b, \mu)\), where \(b\) is a behavior strategy profile and \(\mu\) is a system of beliefs.
- An assessment \((b, \mu)\) is sequentially rational if, for each \(i \in N\) and each \(w \in W_i\), \(b_i\) is a best reply of player \(i\) against \((b, \mu)\) at \(w\).
- An assessment \((b, \mu)\) is weakly consistent with Bayes rule if \(\mu\) is derived using Bayesian updating in the path of \(b\).

Weak perfect Bayesian equilibrium

A weak perfect Bayesian equilibrium is an assessment that is sequentially rational and weakly consistent with Bayes rule.
Sequential Equilibrium (Kreps and Wilson 1982)

- An **assessment** is a pair \((b, \mu)\), where \(b\) is a behavior strategy profile and \(\mu\) is a **system of beliefs**
- An assessment \((b, \mu)\) is **sequentially rational** if, for each \(i \in N\) and each \(w \in W_i\), \(b_i\) is a best reply of player \(i\) against \((b, \mu)\) at \(w\).
- An assessment \((b, \mu)\) is **weakly consistent with Bayes rule** if \(\mu\) is derived using Bayesian updating in the path of \(b\)

**Weak perfect Bayesian equilibrium**

A **weak perfect Bayesian equilibrium** is an assessment that is **sequentially rational** and **weakly consistent with Bayes rule**

- Enough to rule out the undesirable behavior of subgame perfection in the previous example
Sequential Equilibrium (Kreps and Wilson 1982)

- An assessment is a pair $(b, \mu)$, where $b$ is a behavior strategy profile and $\mu$ is a system of beliefs.
- An assessment $(b, \mu)$ is **sequentially rational** if, for each $i \in N$ and each $w \in W_i$, $b_i$ is a best reply of player $i$ against $(b, \mu)$ at $w$.
- An assessment $(b, \mu)$ is **weakly consistent with Bayes rule** if $\mu$ is derived using Bayesian updating in the path of $b$.

Weak perfect Bayesian equilibrium

A **weak perfect Bayesian equilibrium** is an assessment that is **sequentially rational** and **weakly consistent with Bayes rule**.

- Enough to rule out the undesirable behavior of subgame perfection in the previous example.
- Too much freedom for the formation of off-path beliefs.
Sequential Equilibrium (Kreps and Wilson 1982)

- An **assessment** is a pair \((b, \mu)\), where \(b\) is a behavior strategy profile and \(\mu\) is a system of beliefs.
- An assessment \((b, \mu)\) is **sequentially rational** if, for each \(i \in N\) and each \(w \in W_i\), \(b_i\) is a best reply of player \(i\) against \((b, \mu)\) at \(w\).
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A **weak perfect Bayesian equilibrium** is an assessment that is **sequentially rational** and **weakly consistent with Bayes rule**.

- Enough to rule out the undesirable behavior of subgame perfection in the previous example
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**Sequential equilibrium**

A **sequential equilibrium** is an assessment that is **sequentially rational** and **consistent**
Sequential Equilibrium (Kreps and Wilson 1982)

- An **assessment** is a pair \((b, \mu)\), where \(b\) is a behavior strategy profile and \(\mu\) is a **system of beliefs**
- An assessment \((b, \mu)\) is **sequentially rational** if, for each \(i \in N\) and each \(w \in W_i\), \(b_i\) is a best reply of player \(i\) against \((b, \mu)\) at \(w\).
- An assessment \((b, \mu)\) is **weakly consistent with Bayes rule** if \(\mu\) is derived using Bayesian updating in the path of \(b\).

**Weak perfect Bayesian equilibrium**

A **weak perfect Bayesian equilibrium** is an assessment that is **sequentially rational** and **weakly consistent with Bayes rule**

- Enough to rule out the undesirable behavior of subgame perfection in the previous example
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Relations between Equilibrium Concepts in Extensive Games
Relations between Equilibrium Concepts in Extensive Games

Behavior strategy profiles

- NASH
- SUBGAME PERFECT
- SEQUENTIAL
- PERFECT in $E(G_A)$
- PROPER in $E(G_A)$
- PERFECT in $E(G_T)$
- PROPER in $E(G_T)$

Limit behavior strategy profiles induced by PROPER in $E(G_T)$
References


Noncooperative Game Theory

Extensive games

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March 15th, 2011
Proposed Tasks I

**Exercise 1.** Present an definition of extensive game that allows for infinite action sets (not necessarily discrete). When particularized to finite games, this definition should be equivalent to Selten’s definition.

- C. Alós-Ferrer and K. Ritzberger (2008), *Trees and extensive forms*. Journal of Economic Theory 143, 216-250. (More general setting in which time can also move continuously)

**Exercise 2.** Discuss the relations between the notions of perfect equilibrium obtained when defined directly on the extensive game or on the associated strategic game.


**Exercise 3.** Discuss the relations between the notions of proper equilibrium obtained when defined directly on the extensive game or on the associated strategic game.

Proposed Tasks II

**Exercise 4.** Show examples that illustrate that most of the equilibrium concepts we have discussed for extensive games (perfect, proper, sequential) are sensible to apparently equivalent representations of the interactive situation under study.


**Exercise 5.** Present a formal model of a (possibly extensive) game with incomplete information and develop the notion of Nash equilibrium for such a model. These games incomplete information games are often called Bayesian games.


**Exercise 6.** The electronic mail game presents a situation where equilibrium behavior crucially depends on the common knowledge of a certain event. Discuss formally this game or, more challenging, discuss an original game that also exhibits a similar property.