Measuring consensus in ordinal settings

José Luis GARCÍA-LAPRESTA

PRESAD Research Group
University of Valladolid, Spain

Workshop Challenges of Mathematics for Games

Seville, March 26, 2011

Introduction

- Each member of a committee arranges a set of alternatives by means of a weak order (complete preorder)

R_1	R_2	R_3
x_1	x_1	$x_1\, x_2$
x_2	x_2	x_3
x_3	$x_3\, x_4$	x_4
x_4		

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- How similar are their opinions?
- Could we measure consensus?
- In some cases it is necessary a minimum consensus in a group for making a decision

Introduction

- **Bosch (2005)** introduced the notion of consensus measure in the context of linear orders
- **García-Lapresta & Pérez-Román (2008)** extended Bosch's concept to the context of weak orders (complete preorders)
- **García-Lapresta & Pérez-Román (2011)** have analyzed some consensus measures based on metrics in the context of weak orders (complete preorders)
- **Alcalde-Unzu & Vorsatz (2011)** have introduced and characterized some consensus measures (in the context of linear orders) related to some rank correlation indices

Notation

- $V = \{v_1, \ldots, v_m\}$ set of agents $\quad m \geq 3$
- $\mathcal{P}_2(V) = \{I \subseteq V \mid |I| \geq 2\}$
- $X = \{x_1, \ldots, x_n\}$ set of alternatives $\quad n \geq 3$

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- A profile is a vector $\boldsymbol{R} = (R_1, \ldots, R_m)$ of weak orders

Consensus measures on weak orders

A **consensus measure** on $W(X)^m$ is a mapping

$$\mathcal{M} : W(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$
$$(\boldsymbol{R}, I) \longmapsto \mathcal{M}(\boldsymbol{R}, I)$$

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that satisfies the following conditions:

- *Unanimity*. For all $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}, I) = 1 \;\Leftrightarrow\; R_i = R_j \;\text{ for all }\; v_i, v_j \in I$$

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- *Anonymity*. For all permutation π on $\{1, \ldots, m\}$, $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I)$$

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- *Neutrality*. For all permutation σ on $\{1, \ldots, n\}$, $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds
$$\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Consensus measures on weak orders

Other properties that a consensus measure may satisfy:

- *Maximum dissension*. For all $\boldsymbol{R} \in W(X)^m$ and $v_i, v_j \in V$ such that $i \neq j$ it holds

$$\mathcal{M}(\boldsymbol{R}, \{v_i, v_j\}) = 0 \;\Leftrightarrow\; R_i, R_j \in L(X) \text{ and } R_j = R_i^{-1}$$

Consensus measures on weak orders

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- *Reciprocity*. For all $\boldsymbol{R} \in W(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}^{-1}, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Distances and metrics

- A **distance** on a set $A \neq \emptyset$ is a mapping $d : A \times A \longrightarrow \mathbb{R}$ satisfying the following conditions for all $a, b \in A$:
  1. *Non-negativity*. $d(a, b) \geq 0$
  2. *Symmetry*. $d(a, b) = d(b, a)$
  3. *Reflexivity*. $d(a, a) = 0$

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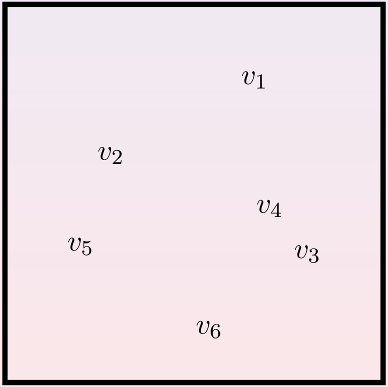
  3. *Reflexivity*. $d(a,a) = 0$

- If d satisfies the following additional conditions for all $a, b, c \in A$:
  4. *Identity of indiscernibles*. $d(a,b) = 0 \Leftrightarrow a = b$
  5. *Triangle inequality*. $d(a,b) \leq d(a,c) + d(c,b)$

  then we say that d is a **metric**

Consensus measures based on distances

$$d : W(X) \times W(X) \longrightarrow \mathbb{R} \text{ distance}$$



$$\mathcal{M}_d\left(\boldsymbol{R}, I\right) = 1 - \frac{\displaystyle\sum_{\substack{v_i, v_j \in I \\ i<j}} d(R_i, R_j)}{\displaystyle\binom{|I|}{2} \cdot \Delta_n}$$

$$\Delta_n = \max\{d(R_i, R_j) \mid R_i, R_j \in L(X)\}$$

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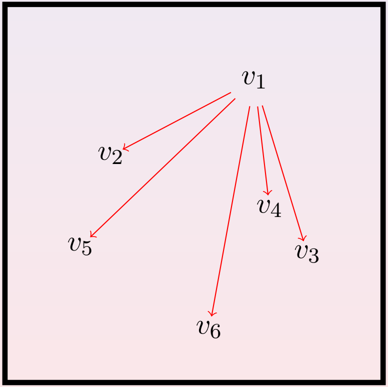

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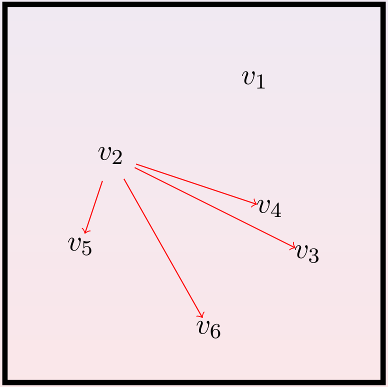

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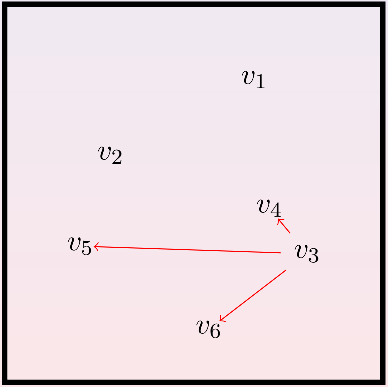

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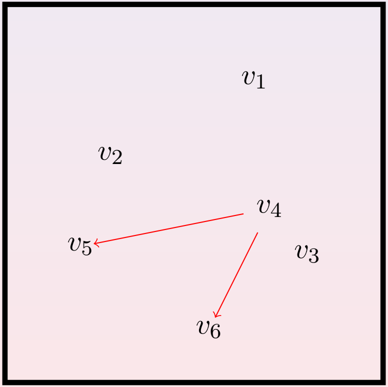

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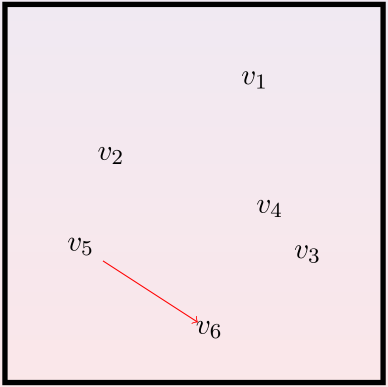

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Consensus measures based on distances

$d : W(X) \times W(X) \longrightarrow \mathbb{R}$ distance

- d is *neutral* if for every permutation σ on the set of alternatives and all $R_1, R_2 \in W(X)$ it holds

$$d\left(R_1^\sigma, R_2^\sigma\right) = d(R_1, R_2)$$

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$$d\left(R_1^{\sigma}, R_2^{\sigma}\right) = d(R_1, R_2)$$

- d satisfies *identity of indiscernibles* if for all $R_1, R_2 \in W(X)$ it holds

$$d(R_1, R_2) = 0 \Leftrightarrow R_1 = R_2$$

Proposition

If $d : W(X) \times W(X) \longrightarrow \mathbb{R}$ is a neutral distance that satisfies identity of indiscernibles, then $\mathcal{M}_d$ is a consensus measure

Consensus measures based on distances

García-Lapresta & Pérez-Román (2011): "Measuring consensus in weak orders". *Consensual Processes*, Springer

Introduction and analysis of

- $\mathcal{M}_d$ where d is induced by a metric on $\mathbb{R}^m$ (discrete

$$d'((a_1, \ldots, a_n), (b_1, \ldots, b_n)) =$$

$$\begin{cases} 1 & \text{if } (a_1, \ldots, a_n) \neq (b_1, \ldots, b_n) \\ 0 & \text{if } (a_1, \ldots, a_n) = (b_1, \ldots, b_n) \end{cases}$$

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Introduction and analysis of

- $\mathcal{M}_d$ where d is induced by a metric on $\mathbb{R}^m$ (discrete, Manhattan

$$d_1((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sum_{i=1}^{n} |a_i - b_i|$$

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Introduction and analysis of

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$$d_2((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \left(\sum_{i=1}^{n} |a_i - b_i|^2 \right)^{\frac{1}{2}}$$

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$$d_\infty((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \max\left\{|a_1-b_1|,\ldots,|a_n-b_n|\right\}$$

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Introduction and analysis of

- $\mathcal{M}_d$ where d is induced by a metric on $\mathbb{R}^m$
  (discrete, Manhattan, Euclidean, Chebyshev, cosine

$$d_c((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = 1 - \frac{\displaystyle\sum_{i=1}^{n} a_i\, b_i}{\sqrt{\displaystyle\sum_{i=1}^{n} a_i^2}\ \sqrt{\displaystyle\sum_{i=1}^{n} b_i^2}}$$

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$$d_H((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \left(\sum_{i=1}^{n}\left(\sqrt{a_i}-\sqrt{b_i}\right)^2\right)^{\frac{1}{2}}$$

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Introduction and analysis of

- $\mathcal{M}_d$ where d is induced by a metric on $\mathbb{R}^m$ (discrete, Manhattan, Euclidean, Chebyshev, cosine, Hellinger)
- $\mathcal{M}_{d_K}$ where d_K is the Kemeny metric

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consensus measures generated by **weighted Kemeny distances**

The Kemeny metric

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- The Kemeny metric has been extended to weak orders $d_K : W(X) \times W(X) \longrightarrow \mathbb{R}$ as the cardinality of the symmetric difference between the weak orders:

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$$d_K(R_1, R_2) = d_K(R_1, R_3) = 1\,,\; d_K(R_2, R_3) = 2$$

Weighted Kemeny distances

García-Lapresta & Pérez-Román (2010): "Consensus measures generated by weighted Kemeny distances on weak orders". *ISDA'10*, Cairo

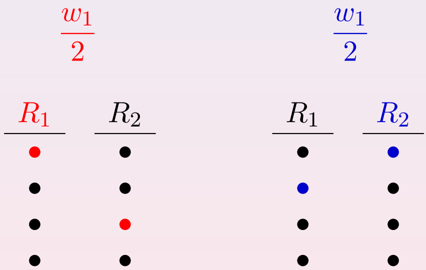
Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n} w_i = 1$

The **weighted Kemeny distance on $W(X)$ associated with w** is the mapping $d_{K,\boldsymbol{w}} : W(X) \times W(X) \longrightarrow \mathbb{R}$ defined by

$$d_{K,\boldsymbol{w}}(R_1, R_2) =$$
$$\frac{1}{2}\Bigg[\sum_{\substack{i,j=1 \\ i<j}}^{n} w_i \left| \text{sgn}\ (o_{R_1}(x_i)^{\sigma_1} - o_{R_1}(x_j)^{\sigma_1}) - \text{sgn}\ (o_{R_2}(x_i)^{\sigma_1} - o_{R_2}(x_j)^{\sigma_1}) \right|$$
$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n} w_i \left| \text{sgn}\ (o_{R_2}(x_i)^{\sigma_2} - o_{R_2}(x_j)^{\sigma_2}) - \text{sgn}\ (o_{R_1}(x_i)^{\sigma_2} - o_{R_1}(x_j)^{\sigma_2}) \right| \Bigg]$$

Weighted Kemeny distances

$$\boldsymbol{w} = (w_1, w_2, w_3)$$



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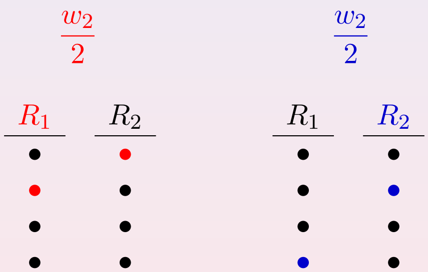

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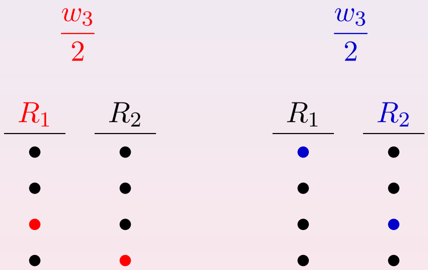

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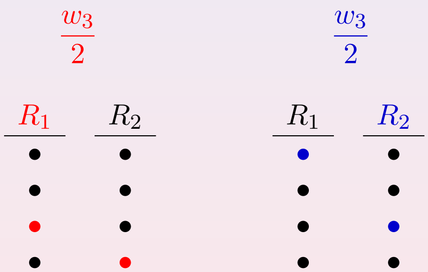

- $d_{K,\boldsymbol{w}}$ is a neutral distance on $W(X)$
- $w_1 = \cdots = w_{n-1} = \frac{1}{n-1} \;\Rightarrow\; d_{K,\boldsymbol{w}} = \frac{1}{n-1}\, d_K$

Weighted Kemeny distances: Example

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$$\boldsymbol{w} = (\tfrac{3}{6}, \tfrac{2}{6}, \tfrac{1}{6})$$

$$d_K(R_1, R_2) = 1 \qquad d_K(R_1, R_3) = 1 \qquad d_K(R_2, R_3) = 2$$

$$d_{K,\boldsymbol{w}}(R_1, R_2) = \tfrac{1}{6}$$

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Results

Let $\boldsymbol{w} = (w_1, \ldots, w_{n-1}) \in [0,1]^{n-1}$ be a weighting vector such that $w_1 \geq \cdots \geq w_{n-1}$ and $\sum_{i=1}^{n} w_i = 1$

Consensus measures based on distances

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1. $\mathcal{M}_{d_{K,\boldsymbol{w}}}$ satisfies anonymity and neutrality
2. $\mathcal{M}_{d_{K,\boldsymbol{w}}}$ satisfies unanimity $\Leftrightarrow$ $w_{n-1} > 0$
3. $\mathcal{M}_{d_{K,\boldsymbol{w}}}$ satisfies maximum dissension $\Leftrightarrow$ $w_{\lfloor \frac{n+1}{2} \rfloor} > 0$

Consensus measures based on distances

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5. $\mathcal{M}_{d_{K,\boldsymbol{w}}}$ is reciprocal $\Leftrightarrow$ $w_1 = \cdots = w_{n-1} = \frac{1}{n-1}$

Preference-approval structures

- Brams (2008): *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures*, Princeton University Press
- Brams & Sanver (2009): "Voting systems that combine approval and preference". *The Mathematics of Preference, Choice and Order: Essays in Honour of Peter C. Fishburn*, Springer-Verlag

$$
\begin{array}{c} x_2 \\ x_3 \\ x_5 \\ x_1 \\ x_4 \\ \hline x_7 \\ x_6 \end{array}
\quad \neq \quad
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Preference-approval structures

Extension to the context of weak orders (complete preorders)

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A preference-approval on X is a pair $(R, G) \in W(X) \times \mathcal{P}(X)$ satisfying the following condition

$$\forall x_i, x_j \in X \; \Big((x_i \, R \, x_j \text{ and } x_j \in G) \; \Rightarrow \; x_i \in G \Big)$$

G is called the set of good alternatives

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The set of preference-approvals on X is denoted by $\mathcal{R}(X)$

A **consensus measure** on $\mathcal{R}(X)^m$ is a mapping

$$\mathcal{M} : \mathcal{R}(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$

that satisfies the following conditions:

- *Unanimity*. For all $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

  $\mathcal{M}(\boldsymbol{R}, I) = 1 \Leftrightarrow (R_i = R_j$ and $G_i = G_j$, for all $v_i, v_j \in I)$

- *Anonymity*. For all permutation π on $\{1, \ldots, m\}$, $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}_\pi, I_\pi) = \mathcal{M}(\boldsymbol{R}, I)$$

- *Neutrality*. For all permutation σ on $\{1, \ldots, n\}$, $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}^\sigma, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Consensus measures on preference-approval structures

Other properties that a consensus measure may satisfy:

- *Maximum dissension*. For all $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $v_i, v_j \in V$ such that $i \neq j$ it holds

$$\mathcal{M}(\boldsymbol{R}, \{v_i, v_j\}) = 0 \;\Leftrightarrow\; (R_i, R_j \in L(X),\; R_j = R_i^{-1} \text{ and } G_j = X \setminus G_i)$$

- *Reciprocity*. For all $\boldsymbol{R} \in \mathcal{R}(X)^m$ and $I \in \mathcal{P}_2(V)$ it holds

$$\mathcal{M}(\boldsymbol{R}^{-1}, I) = \mathcal{M}(\boldsymbol{R}, I)$$

Metrics on preference-approval structures

- The mapping $d_R : \mathcal{R}(X) \times \mathcal{R}(X) \longrightarrow [0,1]$ defined as

$$d_R((R_1, G_1), (R_2, G_2)) = \frac{|(R_1 \cup R_2) \setminus (R_1 \cap R_2)|}{|X \times X| - |X|}$$

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  is a metric

- The convex combination $d = \lambda d_R + (1 - \lambda) d_G$ is a metric for every $\lambda \in (0,1)$

Theorem

The mapping

$$\mathcal{M}_d : \mathcal{R}(X)^m \times \mathcal{P}_2(V) \longrightarrow [0,1]$$

defined as

$$\mathcal{M}_d\,(\boldsymbol{R}, I) = 1 - \frac{\displaystyle\sum_{\substack{v_i, v_j \in I \\ i<j}} d(R_i, R_j)}{\displaystyle\binom{|I|}{2} \cdot \Delta_n}\,,$$

where

$$\Delta_n = \max\left\{d(R_i, R_j) \mid R_i, R_j \in \mathcal{R}(X)\right\}\,,$$

is a consensus measure that satisfies maximum dissension and reciprocity

Measuring consensus in ordinal settings

José Luis GARCÍA-LAPRESTA

PRESAD Research Group
University of Valladolid, Spain

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Seville, March 26, 2011