Asymptotic values of quasiregular maps

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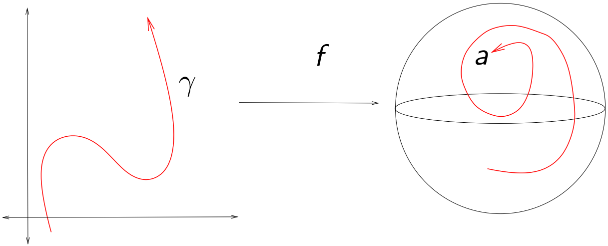
Asymptotic values

$f : \mathbb{R}^d \to \mathbb{R}^m$ continuous, $a \in \mathbb{R}^m \cup \infty$ is an asymptotic value if there exists a continuous path γ, $\gamma \to \infty$, along which,

$$\lim_{\substack{x \to \infty \\ x \in \gamma}} f(x) = a.$$



γ is an asymptotic path and the set of asymptotic values is $\mathrm{As}(f)$.

Holormophic examples

Liouville's Theorem

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic then $\infty \in \mathrm{As}(f)$.

Examples $f : \mathbb{C} \to \mathbb{C}$ holomorphic functions in $\mathbb{C}$.

1. $f(z) = p(z)$ polynomial, $\mathrm{As}(f) = \{\infty\}$.
2. $f(z) = e^z$ exponential, $\mathrm{As}(f) = \{0, \infty\}$.

Question: How large can $\mathrm{As}(f)$ be?

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Question: How large can $\mathrm{As}(f)$ be?

For analytic functions, it depends on how large the function is near ∞ which can be quantified by the order of growth of f, ρ_f.

Order of growth

The order of growth of $f : \mathbb{C} \to \mathbb{C}$ holomorphic,

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r},$$

where $M_f(r) = \max_{|z|=r} |f(z)|$ is the maximum modulus.

Examples

1. $f(z) = p(z)$ polynomial, $\rho_f = 0$.
2. $f(z) = e^z$ exponential, $\rho_f = 1$.

Ahlfors Theorem

Theorem (Denjoy-Carleman-Ahlfors)

If $f : \mathbb{C} \to \mathbb{C}$ holomorphic then

$$\sharp \operatorname{As}(f) \leq 2\rho_f + 1.$$

The "1" corresponds to ∞.

Consequences

- If $\rho_f < \infty$ then $\operatorname{As}(f)$ is finite.
- If $\rho_f < 1/2$ then $\operatorname{As}(f) = \{\infty\}$, i.e. f has no finite asymptotic value.

Infinite order of growth

Ahlfors Theorem characterizes the set of asymptotic values for holomorphic functions with finite ρ_f.

When $\rho_f = \infty$:

Mazurkiewicz 30's

$f : \mathbb{C} \to \mathbb{C}$ holomorphic then $\mathrm{As}(f)$ is an analytic set in the sense of Suslin.

Heins 50's

Given $A \subset \mathbb{C}$ (Suslin) analytic set there exists $f : \mathbb{C} \to \mathbb{C}$ holomorphic such that $\mathrm{As}(f) = A \cup \{\infty\}$.

In Heins' examples, $\rho_f = \infty$.

Analytic sets in $\mathbb{R}^d$

The collection of Borel sets (in $\mathbb{R}^d$) is the smallest σ-algebra that contains all open and closed sets.

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⚠ The continuous image of a Borel set is not necessarily Borel!!

Analytic sets are the images of Borel sets under continuous functions.

Properties of analytic sets

- Every Borel set is analytic.
- Borel sets are those analytic sets whose complement is also analytic.
- Analytic sets are Lebesgue measurable.

Suslin sieve

$A \subset \mathbb{R}^d$ analytic iff there exists a collection of closed sets $\{E_{n_1,\dots,n_k}\}$, labelled with $\{n_1, \dots, n_k\}$, finite sequences of natural numbers, such that

$$A = \bigcup_{\{n_1,n_2,\dots\}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} E_{n_1,\dots,n_k}.$$

(The Suslin operation applied to $\{E_{n_1,\dots,n_k}\}$.)

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Moreover, the sets $\{E_{n_1,\dots,n_k}\}$ can be chosen

i) $E_{n_1,\dots,n_k} \neq \varnothing$ for every $n_1, \dots, n_k$.

ii) $E_{n_1,\dots,n_k,n_{k+1}} \subset E_{n_1,\dots,n_k}$

iii) $\operatorname{diam}(E_{n_1,\dots,n_k}) \leq \delta_k$ for all k and for a given sequence $\delta_k \searrow 0$.

Holormophic functions

$\Omega \subset \mathbb{C}$ open, $f : \Omega \to \mathbb{C}$ is holomorphic iff

1) $f \in C^1(\Omega)$ in the $\mathbb{R}$-sense,
2) $|f'(z)|^2 = J_f(z)$ for every $z \in \Omega$,

where $f'(z)$ is a linear operator, $|\cdot|$ its norm and $J_f(z)$ its determinant.

If f is an homeomorphism f is conformal.

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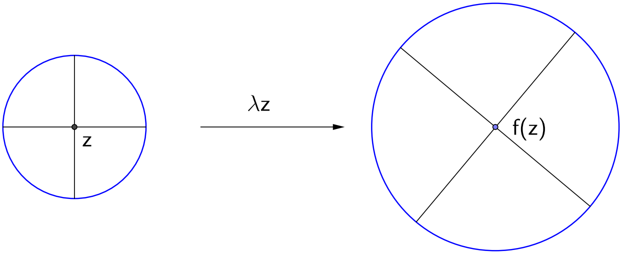
Geometrically: $f(z) = \lambda z + o(z)$ if $f'(z) \neq 0$. In the tangent plane



Maps in $\mathbb{R}^d$

To extend the definition to $\mathbb{R}^d$, $d > 2$:

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Quasiregular maps in $\mathbb{R}^d$

$\Omega \subset \mathbb{R}^d$ open, $f : \Omega \to \mathbb{R}^d$ continuous, $K \geq 1$. f is K-quasiregular in Ω iff

1') $f \in W^{1,d}_{\text{loc}}(\Omega)$ ($\Longrightarrow f'(x)$ exists ae x),

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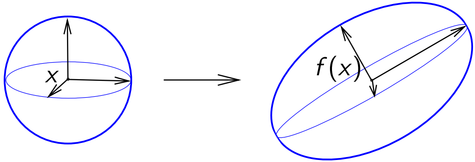
Geometrically: If $f'(x) \neq 0$, in the tangent space



Remarks

- If f is an homeomorphism then f is K-quasiconformal.
- When $d = 2$, f is K-quasiregular iff $f = h \circ g$ with h holomorphic and g K-quasiconformal.
- If $d \geq 3$, f is 1-quasiregular ($f \neq$ ctant) then f is a Möbius transformation.

So the interesting cases are $K > 1$ and $d \geq 3$.

Properties of quasiregular maps

shared with holomorphic functions

$f : \Omega \to \mathbb{R}^d$ quasiregular then,

- f is open ($\Longrightarrow$ the maximum principle holds) and discrete.
- $\dim(B_f) \leq d - 2$ where B_f is the branching set.
- If $f : \mathbb{R}^d \to \mathbb{R}^d$, f is unbounded (Liouville's Theorem).
- Picard's Theorem for quasiregular maps:

Theorem (Rickman)

$f : \mathbb{R}^d \to \mathbb{R}^d$, a non-constant K-quasiregular map. Then f omits at most $q = q(d, K)$ points.

Moreover, for $d \geq 3$, $q(d, K) \nearrow \infty$ as $K \nearrow \infty$.

Order of growth

The order of growth of a qr map is

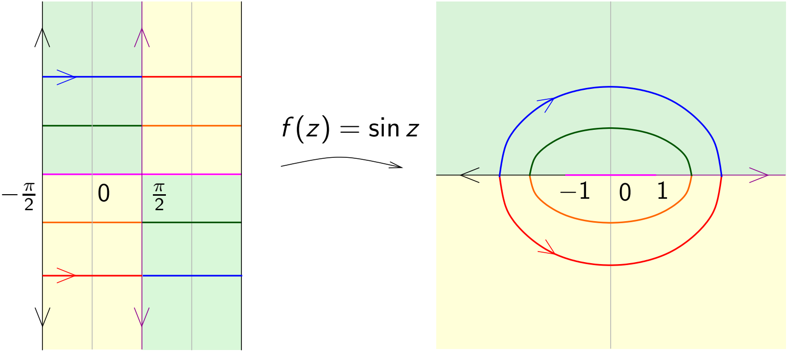
$$\rho_f = \limsup_{r\to\infty} \frac{(d-1)\log\log M_f(r)}{\log r}$$

where $M_f(r) = \sup_{|x|=r} |f(x)|$.

Example of a qr map

Drasin's sine function

It is a higher dimensional analog of the holormophic sine.



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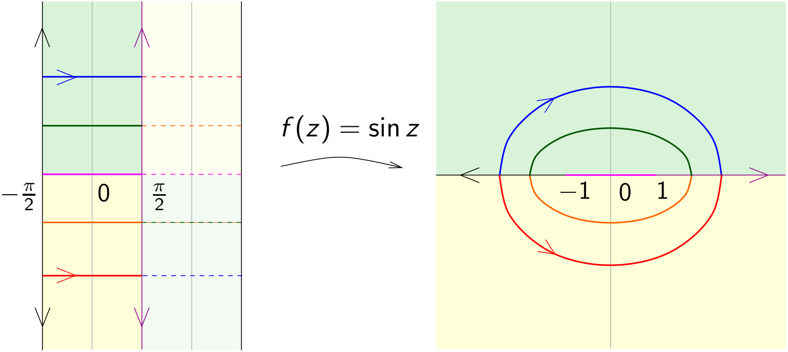
Consider a fundamental domain,

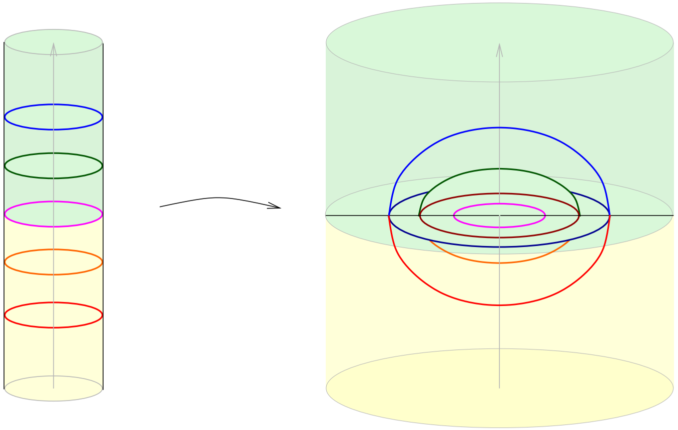


and rotate it around the vertical axis.

Example of a qr map

Drasin's sine function

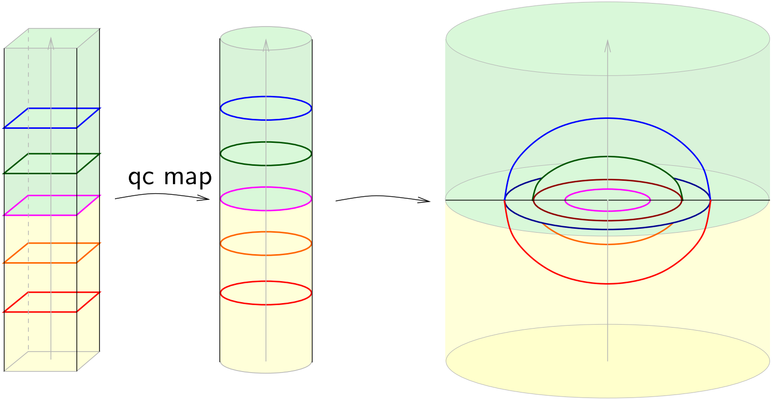
The rotation gives,



Example of a qr map

Drasin's sine function

To extend the function to $\mathbb{R}^d$, map the cylinder onto a square based prism in a bilipschitz way.



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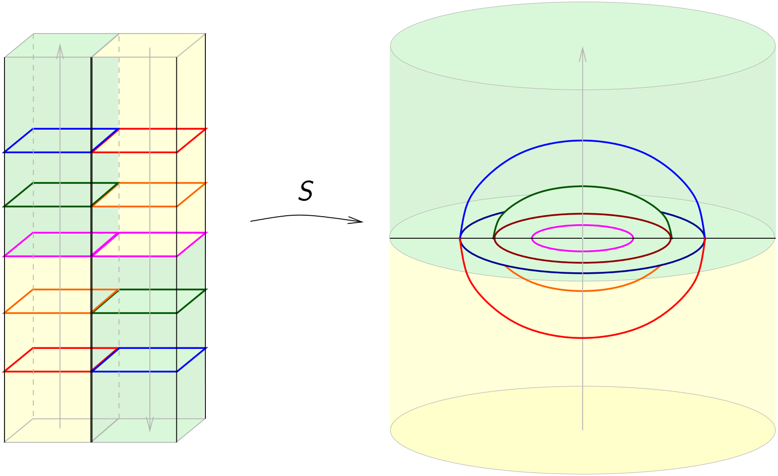
Extend the function by symmetry



Properties of Drasin's sine function

- Periodic on the first $d-1$ variables.
- Bounded on *horizontal* hyperplanes. The faces of the prisms are mapped to $\{y_d = 0\}$.
- Symmetric and orientation preserving.
- Order of growth, $\rho_S = d-1$,
  since $M_S(r) \approx e^{rL}$, where L is the bilispchitz constant.
- $\mathrm{As}(S) = \{\infty\}$,
  the asymptotic paths are eventually contained in one prism with unbounded last coordinate.

Order of growth and asymptotic values

As in Ahlfors Theorem in the holomorphic case:

Theorem (Rickman-Vuorinen)

$f : \mathbb{R}^d \to \mathbb{R}^d$, *K-qr map. If* $\rho_f < c(d, K)$ *then* $As(f) = \{\infty\}$.

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Theorem (Holopainen-Rickman)

For every $d \geq 3$, there exists $f : \mathbb{R}^d \to \mathbb{R}^d$ qr map with $\rho_f < 1$ and infinitely many asymptotic values.

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For every $d \geq 3$, *there exists* $f : \mathbb{R}^d \to \mathbb{R}^d$ *qr map with* $\rho_f < 1$ *and infinitely many asymptotic values.*

Moreover,

Theorem (Drasin)

For every $d \geq 3$, *there exists a qr map* $f : \mathbb{R}^d \to \mathbb{R}^d$ *with* $\rho_f = d - 1$ *and* $As(f) = \mathbb{R}^d \cup \{\infty\}$.

Asymptotic values and analytic sets

Theorem (Mazurkiewicz, C-Qu)

If $f : \mathbb{R}^d \to \mathbb{R}^d$ is qr, then $As(f)$ is an analytic set that contains ∞.

Theorem (C-Qu)

For $d \geq 3$ and any analytic set $A \subset \mathbb{R}^d$ there exists a qr map $f : \mathbb{R}^d \to \mathbb{R}^d$ with $\rho_f = d - 1$ and $As(f) = A \cup \{\infty\}$.

Comments on the proofs

- The first statement is already contained in Mazurkiewicz's work although not explicitly stated. We give an alternative proof, which is "dual" to that of Mazurkiewicz's.

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Comments on the proofs

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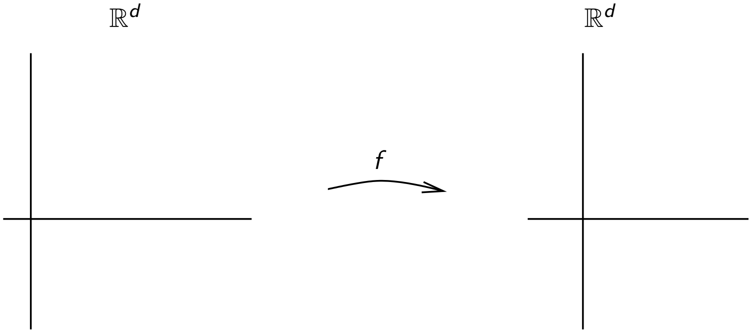
- The proof of the second theorem is based on a modification of Drasin's arguments.
- Main difference with Drasin's work: more control on the asymptotic paths to show that no other asymptotic values are attained.

Proof of the first result

As(f) is an analytic set

f qr, hence discrete and open.



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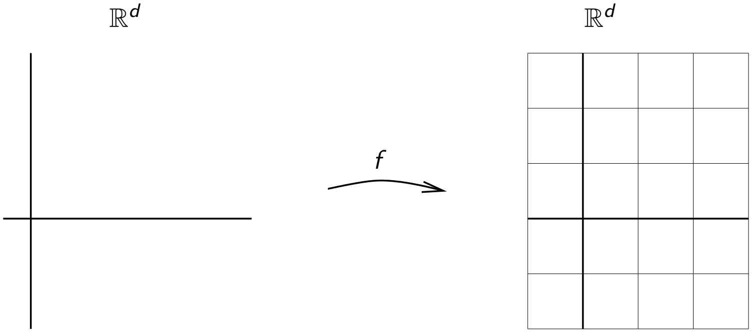

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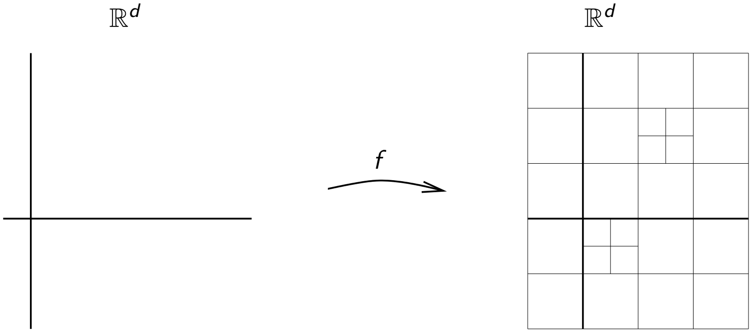

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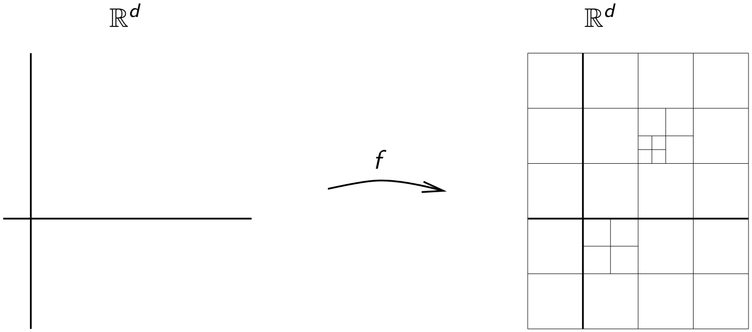

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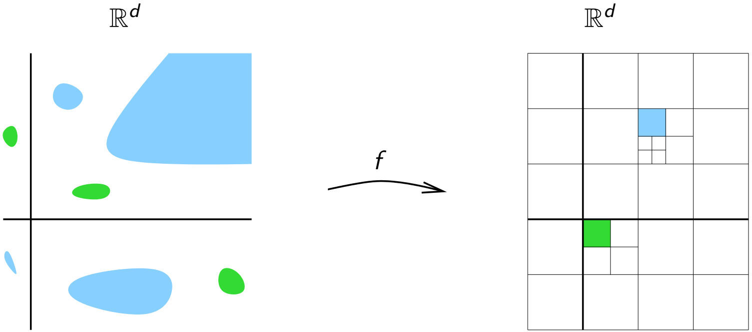
$X_{n_1,\ldots,n_k}$ admissible iff $f^{-1}(X_{n_1,\ldots,n_k})$ has an unbounded connected component.



Proof of the first result

As(f) is an analytic set

Let

$$S_{n_1,\ldots,n_k} = \begin{cases} X_{n_1,\ldots,n_k}, & X_{n_1,\ldots,n_k} \text{ admissible,} \\ \varnothing, & \text{otherwise,} \end{cases}$$

and define

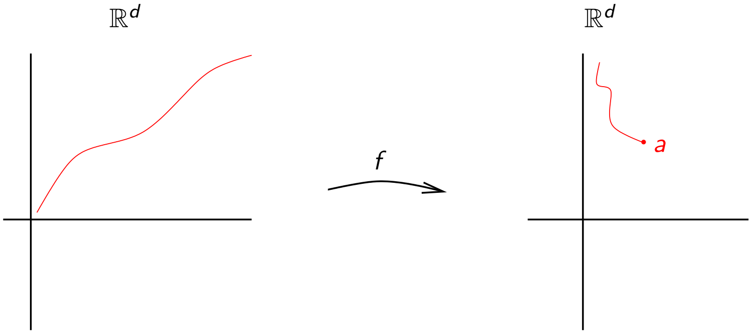
$$B = \bigcup_{\{n_1,n_2,\ldots\}\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} S_{n_1,\ldots,n_k}.$$

Claim

$$B = \mathrm{As}(f) \setminus \{\infty\}.$$

Reason

$a \in \mathrm{As}(f) \setminus \{\infty\}$,



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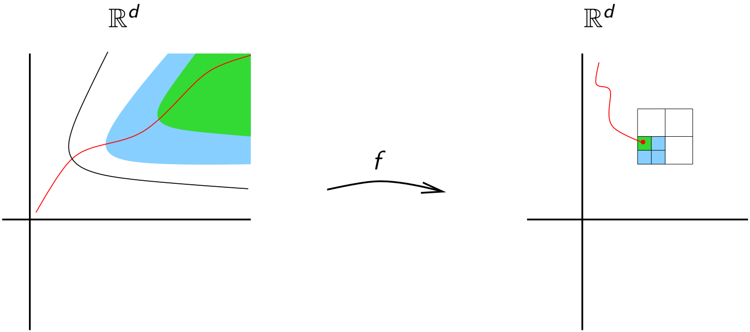

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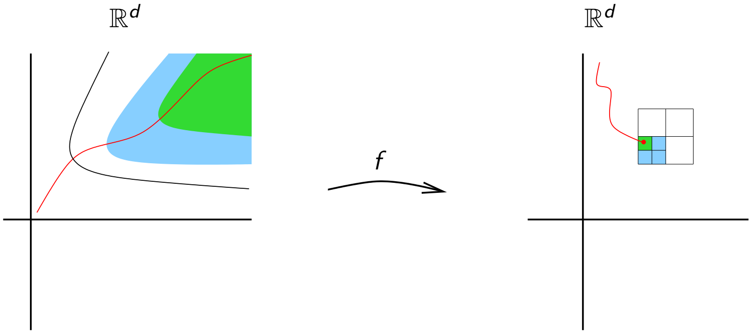
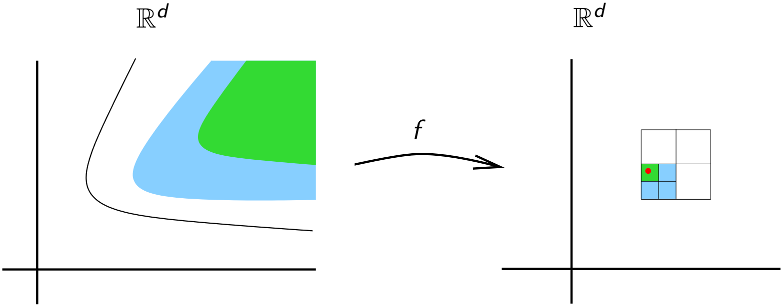


$$a = \bigcap_{k \geq 1} X_{n_1,\ldots,n_k} = \bigcap_{k \geq 1} S_{n_1,\ldots,n_k} \in B.$$

Converse

$b \in B \in \mathbb{R}^d \Longrightarrow b = \bigcap_{k \geq 1} X_{n_1,\dots,n_k}, \quad X_{n_1,\dots,n_k}$ admissible.



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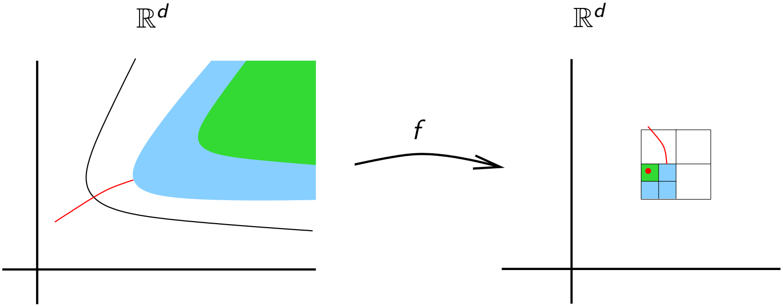

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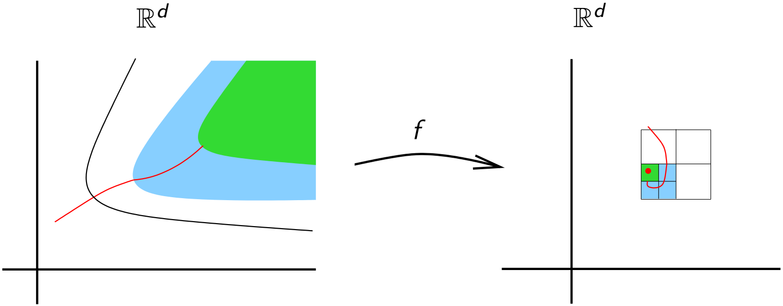

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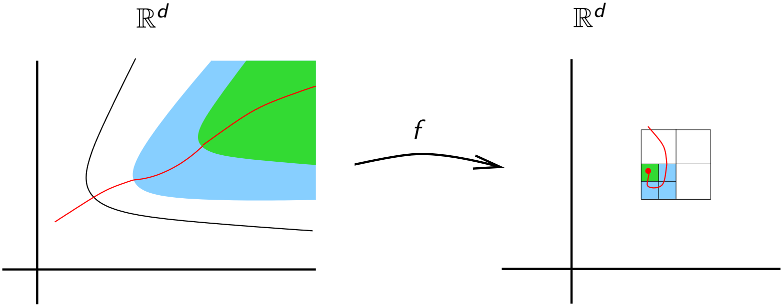

$\gamma \in \mathbb{R}^d$, $\gamma \to \infty$ and

$$\lim_{\substack{x \to \infty \\ x \in \gamma}} f(x) = b \Longrightarrow b \in \mathrm{As}(f) \setminus \{\infty\}.$$