Control Theorems in characteristic $p > 0$ for non commutative Iwasawa Theory

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Universidad de Sevilla

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Control Theorem [Mazur,1972]

Let $p \in \mathbb{Z}$ be a prime of good ordinary reduction for E. For every finite extension F_n/F contained in F_∞ the maps

$$Sel_E(F_n)_p \to Sel_E(F_\infty)_p^{\mathrm{Gal}(F_\infty/F_n)}$$

have finite kernels and cokernels of order bounded independently of n. Moreover, $Sel_E(F_\infty)_p^\vee$ is a finitely generated Λ-module. If $Sel_E(F)_p$ is finite $Sel_E(F_\infty)_p^\vee$ is a torsion module.

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Goal

Structure of $Sel_A(K)_\ell^\vee$ as $\Lambda(G)$ and as $\Lambda(H)$-module (leading to a characteristic element).

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Let G be a compact ℓ-adic Lie group without elements of order ℓ and suppose that $\exists\ H \lhd_c G$ such that $G/H \simeq \mathbb{Z}_\ell$. Let M be a finitely generated $\Lambda(G)$-module. If $M \in \mathcal{M}_H(G)$, then M has a characteristic element.

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1 $Sel_A(K)_\ell^\vee$ is finitely generated over $\Lambda(G)$;

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Let G be a compact ℓ-adic Lie group without elements of order ℓ and suppose that $\exists\ H \lhd_c G$ such that $G/H \simeq \mathbb{Z}_\ell$. Let M be a finitely generated $\Lambda(G)$-module. If $M \in \mathcal{M}_H(G)$, then M has a characteristic element.

Strategy of the proof:

1. $Sel_A(K)_\ell^\vee$ is finitely generated over $\Lambda(G)$;
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J. Coates, T. Fukaya, K. Kato, R. Sujatha, O. Venjakob
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Main tool:

Generalization of Mazur's Control theorem

Theorem 1 (Bandini, V.)

For every finite extension F' of F contained in K, let us consider the map

$$a_{K/F'} : Sel_A(F')_\ell \to Sel_A(K)_\ell^{\mathrm{Gal}(K/F')}.$$

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If ramified and bad reduction primes have decomposition groups opens in G, $Ker(a_{K/F'})$ *and* $Coker(a_{K/F'})$ *are bounded independently of* F' *;*

If $|A[p^\infty](K)| < \infty$*, primes in* S *are of good reduction and have inertia groups open in their decomposition groups, then* $|Ker(a_{K/F'})|, |Coker(a_{K/F'})| < \infty$*.*

Proof.

$$\begin{array}{ccccc}
Sel_A(F')_\ell & \hookrightarrow & H^1_{fl}(\mathcal{X}_{F'}, A[\ell^\infty]) & \twoheadrightarrow & \mathcal{G}_A(F') \\
\downarrow a_{K/F'} & & \downarrow b_{K/F'} & & \downarrow c_{K/F'} \\
Sel_A(K)_\ell^{\mathrm{Gal}(K/F')} & \hookrightarrow & H^1_{fl}(\mathcal{X}_K, A[\ell^\infty])^{\mathrm{Gal}(K/F')} & \longrightarrow & \mathcal{G}_A(K)^{\mathrm{Gal}(K/F')}
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Corollary 1

In the setting of Theorem 1 $Sel_A(K)_\ell^\vee$ *is a finitely generated* $\Lambda(G)$*-module.*

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In the setting of Theorem 1 $Sel_A(K)_\ell^\vee$ *is a finitely generated* $\Lambda(G)$*-module.*

Theorem 2 (Bandini, V.)

Let $K = F(A[\ell^\infty])$ $(\ell \neq p)$*, then the kernels and cokernels of the maps*

$$a_{K/F'} : Sel_A(F')_\ell \to Sel_A(K)_\ell^{\mathrm{Gal}(K/F')}$$

are finite for every finite extension F' *of* F *contained in* K.

Theorem 3 (Bandini, V.)

Suppose that $\exists\ H \lhd_c G$ *such that* $G/H \simeq \mathbb{Z}_\ell$, *put* $K' = K^H$ *and consider the map*

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Corollary 2

In the setting of Theorem 3, if $Sel_A(K')_\ell$ *is a cofinitely generated* $\mathbb{Z}_\ell$*-module, then* $Sel_A(K)_\ell^\vee$ *is finitely generated over* $\Lambda(H)$.

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- $S = \{$places of bad reduction for $A\}$ (Serre-Tate, 1968) $\Longrightarrow$ $Sel_A(K)_\ell^\vee$ is always fin. gen. over $\Lambda(G)$;
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- $\ell > 2g + 1 \Longrightarrow G$ has no elements of order ℓ;
- By Weil-pairing $\exists\ H$ such that $\mathrm{Gal}(G/H) \simeq \mathbb{Z}_\ell$; $\Longrightarrow$ If $Sel_A(K^H)_\ell^\vee$ is fin. gen. over $\mathbb{Z}_\ell$ then $Sel_A(K)_\ell^\vee$ is fin. gen. over $\Lambda(H)$.

Thanks!