Computation of Universal Deformation Rings

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Notation

- p rational prime;
- k finite field of characteristic p;
- Ar category of local artinian complete $W(k)$-algebras A with surjective homomorphism $\pi_A : A \to k$ and maximal ideal m_A;
- $\hat{Ar}$ category of local noetherian complete $W(k)$-algebras A with surjective homomorphism $\pi_A : A \to k$ and maximal ideal m_A;
- S finite set of primes of $\mathbb{Q}$ including p and the infinite archimedian prime;
- G_S the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S;

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Deformations

Let $\bar{\rho} : G_S \to GL_n(k)$ be a Galois representation and $A \in Ar$. A lift of $\bar{\rho}$ to A is a representation $\rho_A : G_S \to GL_n(A)$ such that the diagram

$$\begin{array}{ccc} & & GL_n(A) \\ & \overset{\rho_A}{\nearrow} & \downarrow \pi_A \\ G_S & \xrightarrow{\bar{\rho}} & GL_n(k) \end{array}$$

commutes.

We say that two lifts ρ_1, ρ_2 of $\bar{\rho}$ to A are equivalent if there exists $M \in Ker(\pi_A)$ such that $M\rho_1(g)M^{-1} = \rho_2(g)$.

A deformation of $\bar{\rho}$ to A is an equivalence class of lifts.

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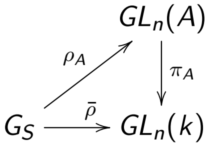

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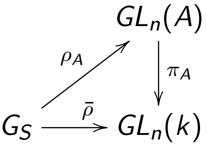

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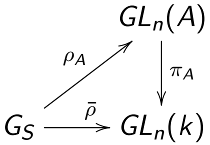

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A deformation of $\bar{\rho}$ to A is an equivalence class of lifts.

Theorem (Mazur)

Suppose that the centralizer of the image of $\bar{\rho}$ is given by the set of scalar matrices. Then there exists a ring $R \in \hat{A}r$ and a deformation $\rho_{univ} : G_S \to GL_n(R)$ such that, for every ring $A \in Ar$ and every deformation $\rho_A : G_S \to GL_n(A)$, there is a unique homomorphism $h : R \to A$ that makes the following diagram commute:

R is called the universal deformation ring associated to $\bar{\rho}$.

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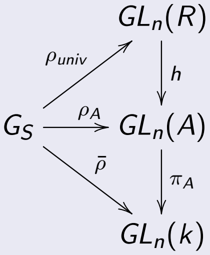

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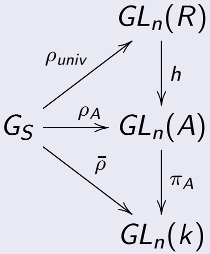

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Let $V_{\bar{\rho}}$ be the $k[G]$-module associated to $\bar{\rho}$.

Definition

Let $A \in Ar$. A deformation of $V_{\bar{\rho}}$ to A is a pair (V_A, ι_A), where

- V_A is a free $A[G]$-module provided with a G-continous action;
- $\iota_A : V_A \otimes_A k \simeq V_{\bar{\rho}}$.

Definition

Let β be a k-basis of $V_{\bar{\rho}}$. A framed deformation of the pair $(V_{\bar{\rho}}, \beta)$ to A is a triple (V_A, ι_A, β_A), where

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Deformation functors

Definition

The deformation functor $F_{\bar{\rho}} : Ar \to Sets$ attached to $\bar{\rho}$ is defined as

$$F_{\bar{\rho}}(A) = \{\text{deformations of } \bar{\rho} \text{ to } A\} \quad (1)$$

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Representability

Theorem (Mazur)

- $F^{\square}_{\bar\rho}$ *is pro-representable, that is, there exists* $R^{\square} = R^{\square}_{\bar\rho} \in \hat{Ar}$ *such that*
$$F^{\square}_{\bar\rho}(A) = Hom_{W(k)}(R^{\square}, A); \tag{3}$$
- *If* $End_G(V_{\bar\rho}) = k$*, then* $F_{\bar\rho}$ *is pro-representable by a ring* $R = R_{\bar\rho} \in \hat{Ar}$*.*

The representing algebras R and $R^{\square}$ are called the universal deformation ring and the universal framed deformation ring attached to $\bar\rho$ respectively.

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First example

Let $p = 3$ and $S = \{3, 7, \infty\}$ and consider the representation

$$\bar{\rho} : G \to GL_2(\mathbb{F}_3) \tag{4}$$

given by the 3-division points of the modular curve $X_0(49)$.
Then $R_{\bar{\rho}} \simeq \mathbb{Z}_3[[x_1, x_2, x_3, x_4]]/((1 + x_4)^3 - 1)$.

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Question: How to compute R in general case?

In 1995 Faltings has described a method to compute a presentation of R of the form

$$W(k)[[X_1, \ldots, X_r]]/(f_1, \ldots, f_t). \quad (5)$$

This presentation is very far from minimal.

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The tangent space

Let ϵ be an element such that $\epsilon^2 = 0$. Then we can define the following.

Definition

Let $F_{\bar{\rho}}$ be a deformation functor. The tangent space of $F_{\bar{\rho}}$ is the set

$$F_{\bar{\rho}}(k[\epsilon]). \qquad (6)$$

It has a natural structure of k-vector space.

Lemma

- $F_{\bar{\rho}}(k[\epsilon]) \simeq H^1(G, Ad(\bar{\rho})) \simeq Ext^1_{k[G]}(V_{\bar{\rho}}, V_{\bar{\rho}})$;
- $dim F^{\square}_{\bar{\rho}}(k[\epsilon]) = dim F_{\bar{\rho}}(k[\epsilon]) + dim Ad(\bar{\rho}) - dim H^0(G, Ad(\bar{\rho}))$.

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Deformation conditions

Let $\mathfrak{P}$ be the category of pairs (A, V_A) with $A \in Ar$ and $V_A \in F_{\bar{\rho}}(A)$. Let $\mathfrak{D}$ be a full subcategory of $\mathfrak{P}$ such that

- if $(A, V_A) \to (B, V_B)$ is a morphism in $\mathfrak{P}$ and $(A, V_A) \in \mathfrak{D}$, then $(B, V_B) \in \mathfrak{D}$.
- $(A \times_C B, V) \in \mathfrak{D} \iff (A, V_A), (B, V_B) \in \mathfrak{D}$.
- if $(A, V_A) \to (B, V_B)$ is a monomorphism in $\mathfrak{P}$ and $(B, V_B) \in \mathfrak{D}$, then $(A, V_A) \in \mathfrak{D}$.

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If $\mathfrak{D}$ is a deformation condition, we can consider the subfunctor $F_{\mathfrak{D}} \subseteq F_{\bar{\rho}}$

$$F_{\mathfrak{D}}(A) = \{\text{deformations } V_A \text{ such that } (A, V_A) \in \mathfrak{D}\} \qquad (7)$$

Lemma

If $F_{\bar{\rho}}$ is representable, then $F_{\mathfrak{D}}$ is represented by a quotient $R_{\mathfrak{D}}$ of R and the tangent space $F_{\mathfrak{D}}(k[\epsilon])$ is a k-vector subspace of $F_{\bar{\rho}}(k[\epsilon])$.

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Galois deformation conditions

Let $\Sigma \subseteq S$. For every prime $\ell \in \Sigma$ we consider the following:

- let G_ℓ be the absolute Galois group of $\mathbb{Q}_\ell$.
- let $\bar{\rho}_\ell = \bar{\rho}|_{G_\ell}$ be the ℓ-adic part of $\bar{\rho}$,
- let $\mathfrak{D}_\ell$ be a deformation condition attached to $\bar{\rho}_\ell$.

We consider the subcategory $\mathfrak{D}$ of pairs (A, V_A) such that $(A, V_A|_{G_\ell}) \in \mathfrak{D}_\ell$ for every $\ell \in \Sigma$. Then $\mathfrak{D}$ is a deformation condition and we call it a global Galois deformation condition.

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- let $\bar{\rho}_\ell = \bar{\rho}|_{G_\ell}$ be the ℓ-adic part of $\bar{\rho}$;
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We consider the subcategory $\mathfrak{D}$ of pairs (A, V_A) such that $(A, V_A|_{G_\ell}) \in \mathfrak{D}_\ell$ for every $\ell \in \Sigma$. Then $\mathfrak{D}$ is a deformation condition and we call it a global Galois deformation condition.

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Minimal ramification

Let $\ell \in \Sigma$ be a prime different from the residual characteristic p and suppose that

$$\bar{\rho}_\ell(I_\ell) \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Let ρ_A be a deformation of $\bar{\rho}$ to A. We say that ρ is minimally ramified at ℓ if

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Lemma

Being minimally ramified is a deformation condition $\mathfrak{D}$. Moreover

$$F_{\mathfrak{D}}(k[\epsilon]) = H^1(G_{\mathbb{F}_\ell}, Ad(\bar{\rho})^{I_\ell}). \quad (10)$$

Flatness

Let ρ_ℓ be a deformation of $\bar{\rho}_\ell$. We say that ρ_ℓ is finite flat (or simply flat) if the representation module V_{ρ_ℓ} , viewed as a finite abelian group with G_ℓ-action, is the $\mathbb{Q}_\ell$-module of $\bar{\mathbb{Q}}_\ell$-points of a finite flat group scheme M over $Spec(\mathbb{Z}_\ell)$.

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A Galois deformation condition $\mathfrak{D} = \{\mathfrak{D}_\ell\}$ is called minimal if

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Kisin's theorem

Suppose that $F_{\bar{\rho}}$ and all of the local functors $F_{\bar{\rho}_\ell}$ are representable. Set

$$R_{loc} = \hat{\otimes}_{\ell \in \Sigma} R_\ell. \tag{11}$$

Moreover let

$$\theta_i : H^i(G, Ad(\bar{\rho})) \to \prod_{\ell \in \Sigma} H^i(G_\ell, Ad(\bar{\rho})) \tag{12}$$

be the restriction map. Set

- $r_i = dim\ Ker(\theta_i)$;
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There exist $f_1, ..., f_{t_1+r_2} \in R_{loc}[[x_1, ..., x_{r_1}]]$ *such that*

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Good points:

- it works for every $\bar{\rho}$;
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- r is NOT an optimal value;
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$R = T$ theorems

Suppose that $\bar{\rho}$ is modular, that is, it comes from the reduction mod p of a p-adic modular form f. Let $\mathfrak{D}$ be a deformation condition for $\bar{\rho}$.

Let $\mathbb{S}_{\mathfrak{D}}$ be the subspace of cusp forms g such that the p-adic representation associated is a deformation of $\bar{\rho}$ of type $\mathfrak{D}$. Let $\mathbb{T}_{\mathfrak{D}}$ be the p-adic completion of the Hecke algebra associated to $\mathbb{S}_{\mathfrak{D}}$.

Theorem (Taylor-Wiles)

If $\mathfrak{D}$ is a minimal condition, then the functorial map

$$R_{\mathfrak{D}} \to \mathbb{T}_{\mathfrak{D}} \tag{14}$$

is an isomorphism. Moreover $R_{\mathfrak{D}}$ is a local complete intersection ring.

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Let E be the elliptic curve given by Weierstrass equation

$$Y^2 + XY = X^3 - X^2 - X - 3 \tag{15}$$

This is curve 142 C 1 in J. Cremona's Tables. It has conductor 142. We take $p = 3$ and $S = \Sigma = \{3, 71\}$. Let

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We consider the local conditions

- $\bar{\rho}$ is flat at 3;
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Let $\mathfrak{D}$ be the resulting deformation condition.
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Theorem (Agashe-Stein)

The ring $\mathbb{T}$ of Hecke operators on cusp forms of weight k and level N is generated as an abelian group by the operators T_n with

$$n \leq \frac{kN}{12} \prod_{p|N} (1 + \frac{1}{p}). \tag{17}$$

Then we only need to compute the vectors $T_n = (a_n(f))_f$ with $n \leq 12$ and f running over the normalised 3-adic eigenforms with Fourier coefficients equal to the ones of $\bar{\rho}$.

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n	$a_n(f_{71})$	$a_n(g_{71})$
1	1	1
2	u	$3-u-u^2$
3	$3-u^2$	$-3+u+u^2$
4	$-2+u^2$	$1+u$
5	$-1-u$	$5-2u-u^2$
6	$3-2u$	$-3-u$
7	$-6+2u+2u^2$	$-6+2u+2u^2$
8	$-3+u$	$-u$
9	$6-3u-u^2$	u
10	$-u-u^2$	$6+u-u^2$
11	$6-2u-2u^2$	$2u$
12	$-6+3u$	$-6+3u-2u^2$

where u is the unique root of the polynomial X^3-5X+3 in $\mathbb{Z}_3[X]$.

Using the approximation $u \equiv 60 \pmod{81}$, we see that $T_{\mathfrak{D}}$ is generated as a $\mathbb{Z}_3$-module by $1 = (1,1)$ and $x = (0,9)$. Then we can conclude

Theorem (Lario-Schoof)

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A non-minimal example

Consider the previous example but taking $S = \{2, 3, 71\}$.

Theorem (Lario-Schoof)

Then $R_{\mathfrak{D}} \simeq \mathbb{Z}_3[[X, Y]]/(f_1, f_2)$, *where*

- $f_1 = 29412Y - 9804Y^2 - 91158XY - 1641XY^2 + 11618X^2Y - 787(X^3 - 15X^2 + 54X)$;
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$R = T$ theorems:

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