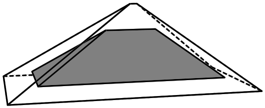
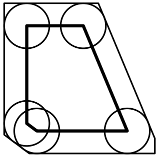
# On decompositions of a polytope using its inner parallel bodies



Eugenia Saorín Gómez

(jointly with Eva Linke)

**OvGU Magdeburg**



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- ▷ $\mathcal{U}(K) = \{u \in S^{n-1} : u \text{ is extreme}\}$.
- ▷ [Krein-Milman-Theorem, 1940]

$$K = \bigcap_{u \in \mathcal{U}(K)} \{x \in \mathbb{R}^n : u^\top x \le h(K, u)\}$$

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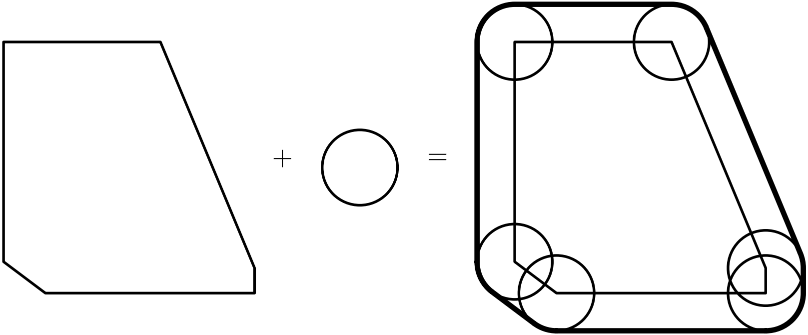

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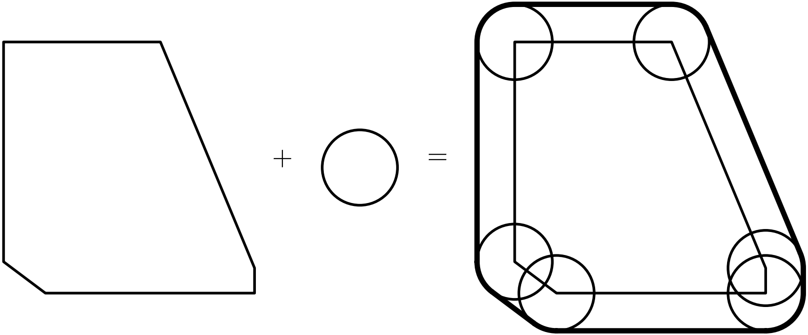

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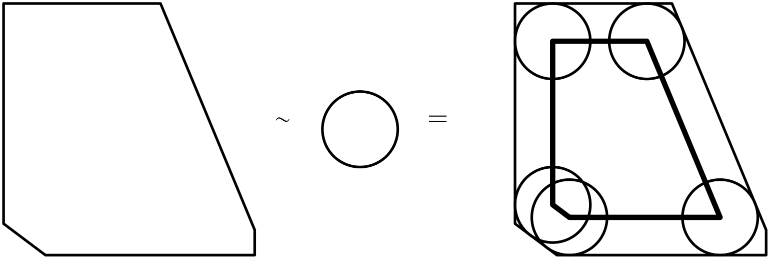

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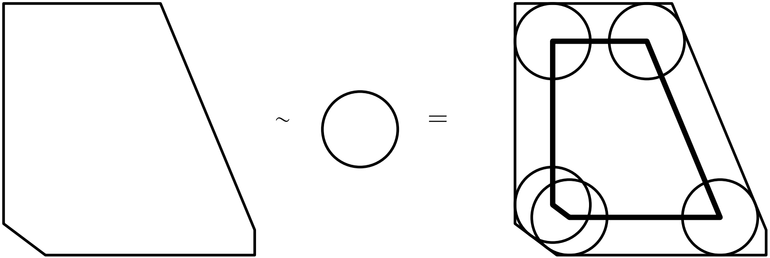

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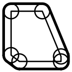
  - $\tau \geq 0$: $K_\tau = K + \tau B^n$ (outer parallel body) 

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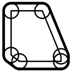
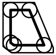
  - $\tau \geq 0$: $K_\tau = K + \tau B^n$ (outer parallel body) 
  - $\tau < 0$: $K_\tau = K \sim |\tau| B^n$ (inner parallel body) 

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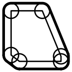
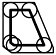
  - $\tau \geq 0$: $K_\tau = K + \tau B^n$ (outer parallel body) 
  - $\tau < 0$: $K_\tau = K \sim |\tau| B^n$ (inner parallel body) 
- It is for $\tau > 0$, $(K_\tau)_{-\tau} = K$, but in general $(K_{-\tau})_\tau \subset K$.

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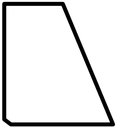

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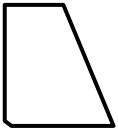

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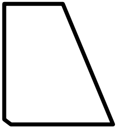
- 0-dimensional faces are called vertices, 1-dimensional faces are called edges, $(\dim(P) - 1)$-dimensional faces are called facets.



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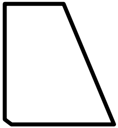

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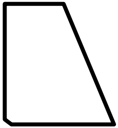

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**Problem.**
Characterize decompositions of $P$ that use the inner parallel body $P_\tau$.

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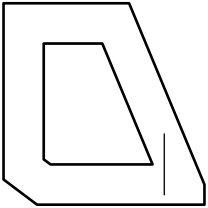

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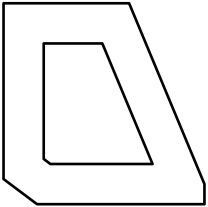

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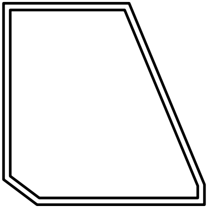

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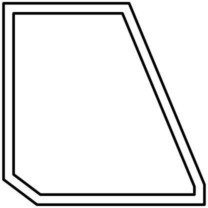

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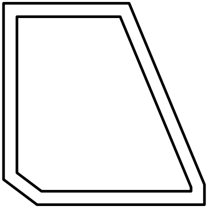

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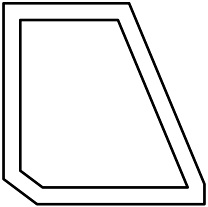

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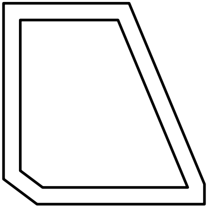

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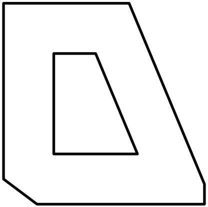

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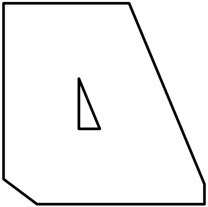

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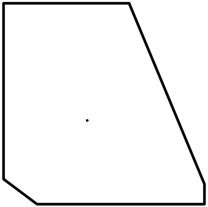

# Inner parallel bodies of polytopes – properties

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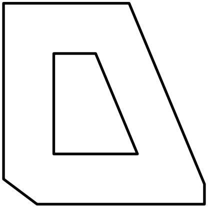
- $P_\tau = \{x \in \mathbb{R}^n : u^\top x \leq h(P, u) + \tau, \text{ for } u \in \mathcal{U}(P)\}$.
- $\mathcal{U}(P_\tau) \subset \mathcal{U}(P)$.
- Let $\tau_0(P) = 0$, $\tau_1(P) = \inf\{\mu \in (-\mathrm{r}(P), 0] : \mathcal{U}(P_\mu) = \mathcal{U}(P)\}$.
- Inductively, let $\tau_i(P) = \tau_1(P_{\tau_{i-1}(P)}) = \inf\{\mu \in (-\mathrm{r}(P), 0] : \mathcal{U}(P_\mu) = \mathcal{U}(P_{\tau_{i-1}(P)})\}$.
- For $u \in \mathcal{U}(P_\tau)$ we have $h(P, u) + \tau = h(P_\tau, u)$.



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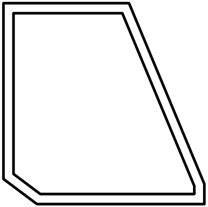

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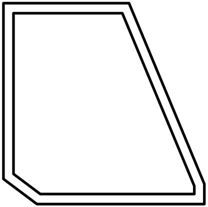
- By continuity, it is also true for $\tau = \tau_1$.



# Situation for $n = 2$

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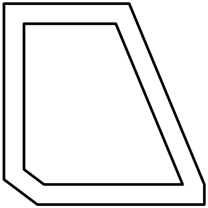

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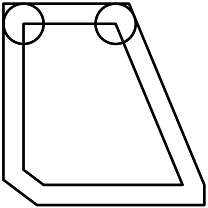
- Let $F(P_\tau, u)$ be an edge of $P_\tau$. Then $F(P, u)$ is an edge of $P$. Furthermore, by definition, $F(P_\tau, u) + |\tau| B_2 \subset P$ and $h(P, u) + \tau = h(P_\tau, u)$.



**Theorem.**
Let $P$ be a convex polygon, let $i \in \mathbb{N}$ with $\tau_{i+1}(P) \leq \tau \leq \tau_i(P)$.

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**Corollary.**
Let $P$ be a convex polygon, $-\mathrm{r}(P) \leq \tau \leq 0$.

$$h(P,u) = h(P_\tau, u) + \int_\tau^0 h((P_\mu)^{\mathbb{1}}, u)\, d\mu \quad \text{for all } u \in S^{n-1}.$$

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**Corollary.**
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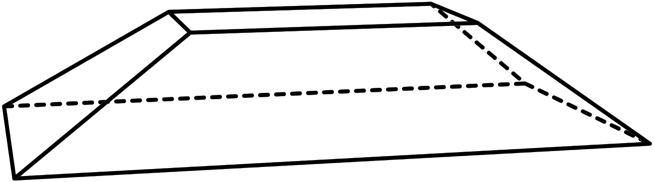

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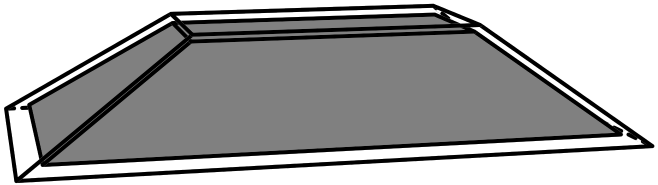

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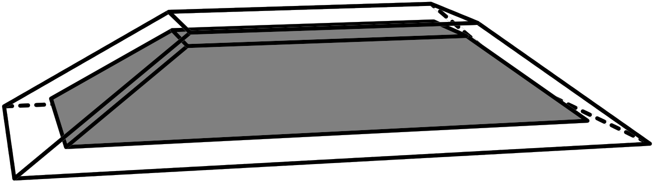

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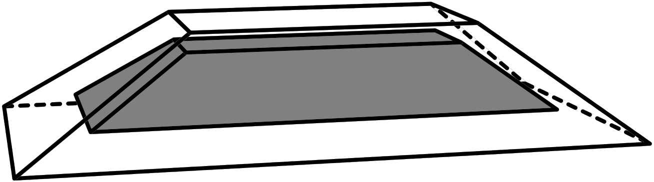

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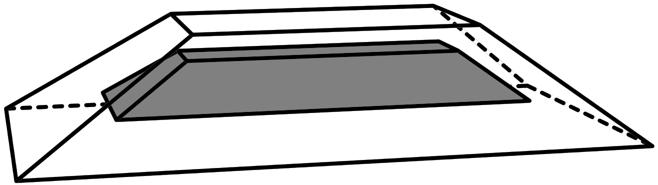

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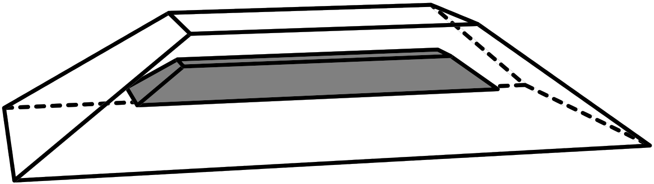

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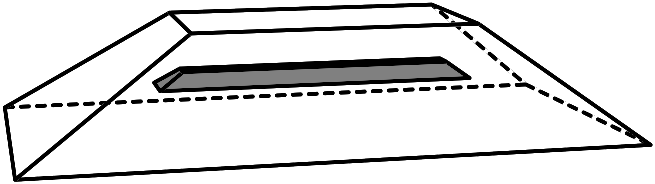

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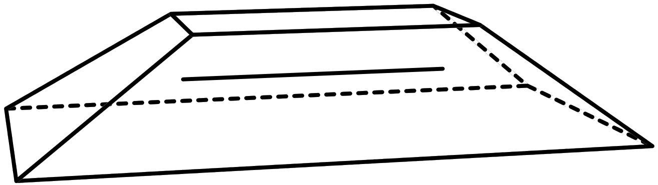

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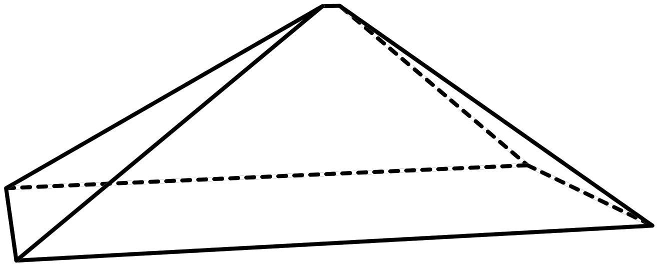

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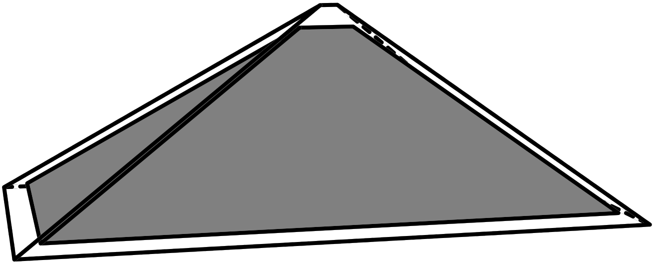

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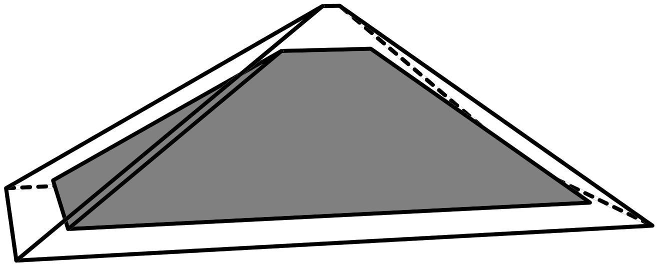

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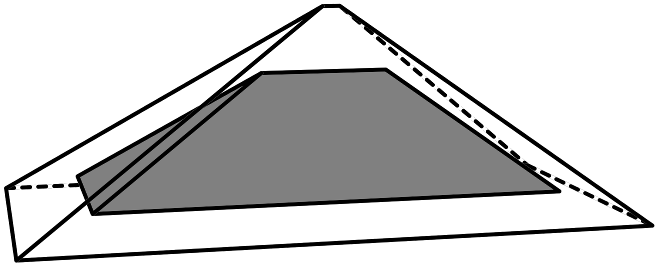

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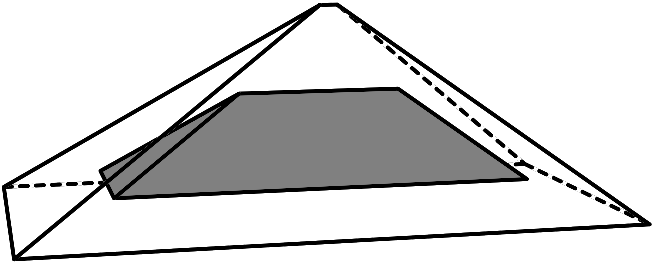

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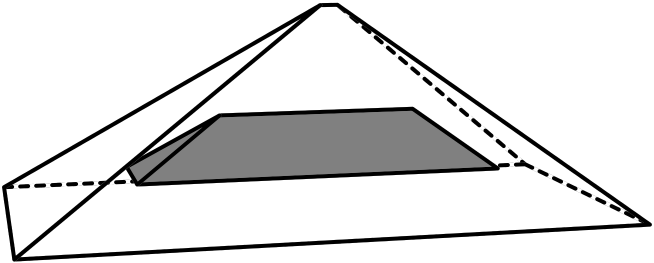

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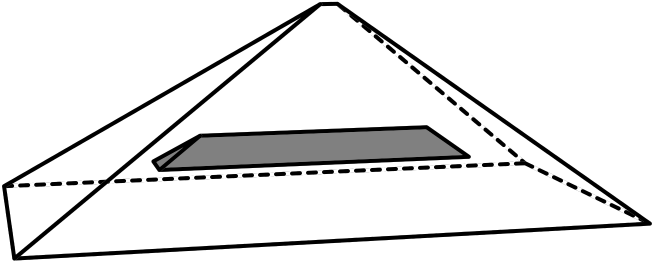

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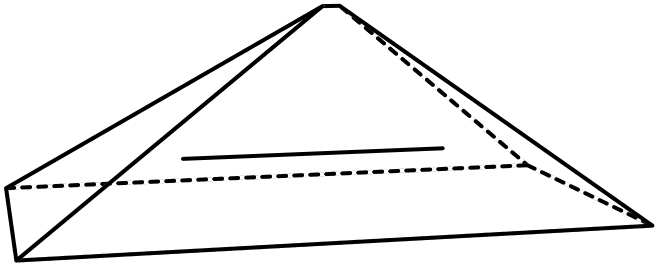

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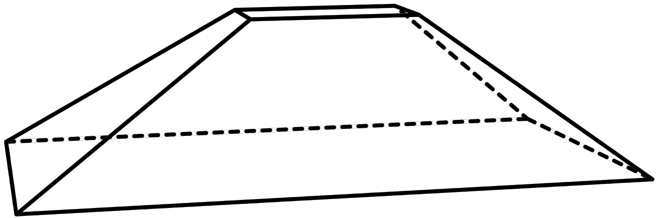

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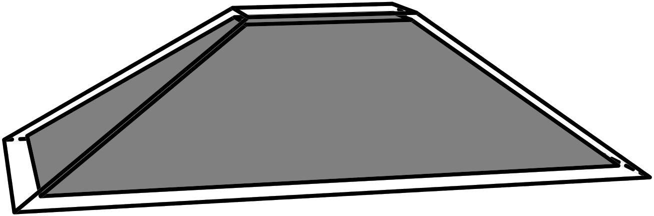

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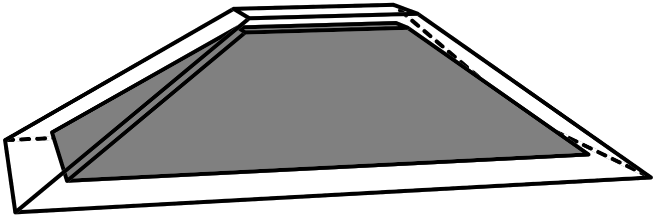

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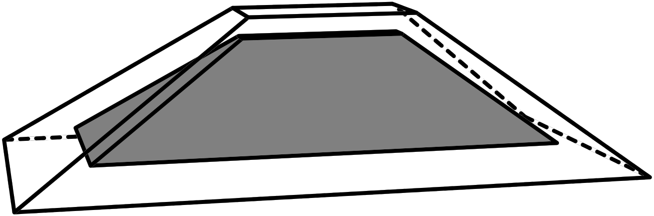

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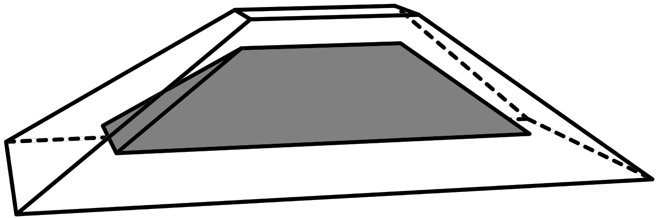

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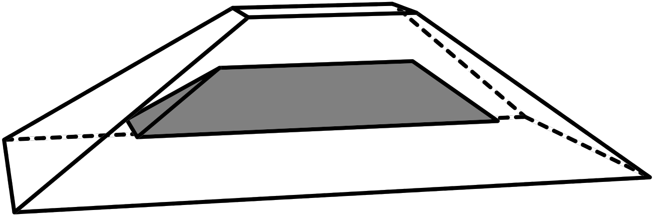

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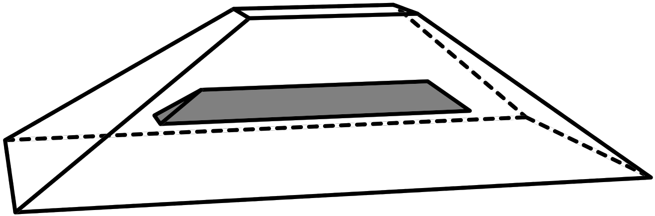

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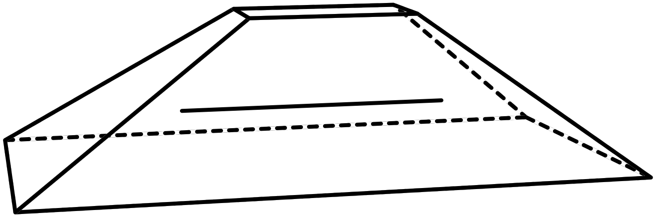

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Let $-\mathrm{r}(K) < \tau < 0$. Then $K_\tau$ is a dilation of $K$ if and only if $K = \mathrm{r}(K)K^{\mathbb{1}}$.

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Let $P$ be indecomposable, $-\mathrm{r}(P) \leq \tau \leq 0$. Then $P_\tau$ is a summand of $P$ if and only if $P = \mathrm{r}(P)P^{\mathbb{1}}$.

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Let $-\mathrm{r}(P) \leq \mu_1 < \mu_2 \leq 0$. The following statements are equivalent:

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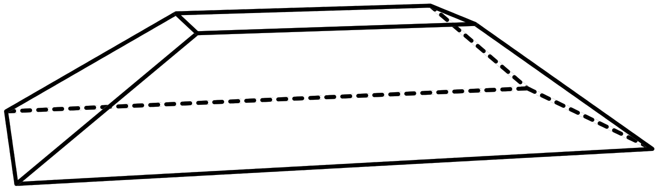

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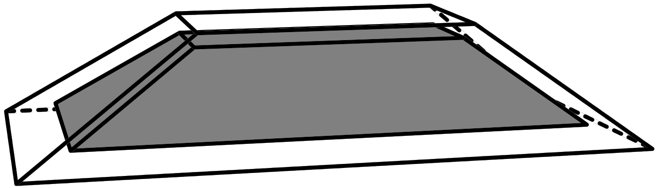

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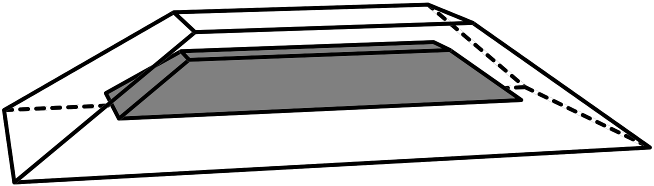

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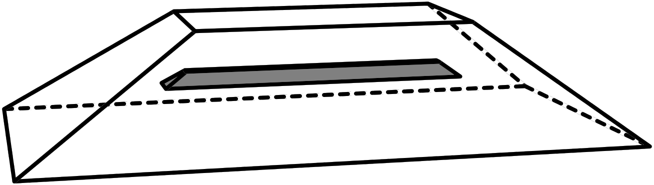

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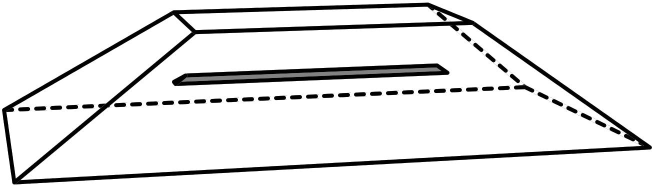

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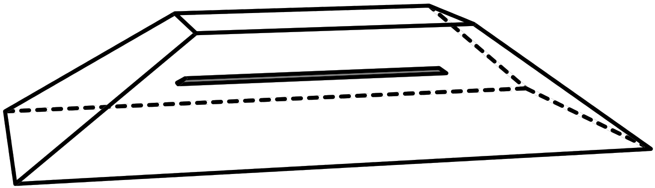

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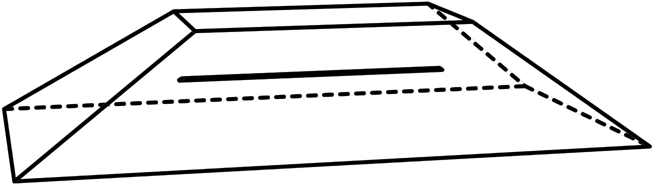

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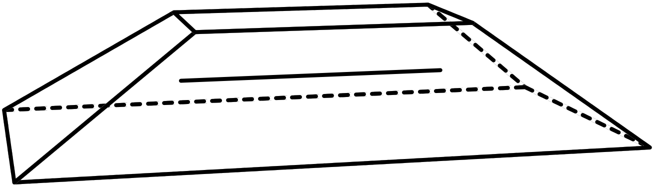

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**Lemma.**
Let $P = P_\tau + |\tau| P^{\mathbb{1}}$ for all $\tau_1(P) \leq \tau \leq 0$. Then $P_\tau = \left(1 - \frac{|\tau|}{|\mu|}\right) P + \frac{|\tau|}{|\mu|} P_\mu$ for all $\tau_1(P) \leq \mu \leq \tau \leq 0$.

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If $P_\tau$ is a summand of $P$ for all $\tau_1(P) \leq \tau \leq 0$ then the normal fans of $P$ and $P_\tau$ are equal for all $\tau_1(P) \leq \tau \leq 0$.

# Proof Ideas

- $P_\tau$ is summand of $P_{\tilde{\tau}}$, for all $\mu_1 \leq \tau \leq \tilde{\tau} \leq \mu_2$.
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and

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Let $-\mathrm{r}(K) \le \tau \le 0$. Then $\frac{d}{d\mu} h(K_\mu, u)|_{\mu=\tau} \ge h((K_\tau)^{\mathbb{1}}, u)$, for all $u \in S^{n-1}$ for which the derivative exists. If $\mathcal{U}(K_\tau + (K_\tau)^{\mathbb{1}}) = \overline{\mathcal{U}(K_\tau)}$, then equality holds.

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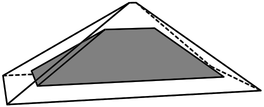
- $\triangleright$ $\frac{d}{d\mu} h(P_\mu, \cdot)|_{\mu=\tau}$ is a support function, for all $\mu_1 \leq \tau \leq \mu_2$ for which the derivative exists.

- $\triangleright$ Let $u \mapsto \frac{d}{d\mu} h(P_\mu, u)|_{\mu=\tau}$ be a support function.
- $\triangleright$ Let $R_\tau$ be polytope with support function $u \mapsto \frac{d}{d\mu} h(P_\mu, u)|_{\mu=\tau}$.
- $\triangleright$ $\mathcal{U}(R_\tau) \subset \mathcal{U}(P_\tau) = \mathcal{U}((P_\tau)^{\mathbb{1}})$.
- $\triangleright$ For $u \in \mathcal{U}(P_\tau)$ it is $h(P_\mu, u) = h(P, u) + \mu$, and thus $\frac{d}{d\mu} h(P_\mu, u)|_{\mu=\tau} = 1 = h((P_\tau)^{\mathbb{1}}, u)$.

# On decompositions of a polytope using its inner parallel bodies



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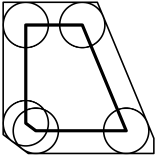

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