Random planar graphs with minimum degree two and three

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(Random planar 2-graphs and 3-graphs)

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Situation

Labelled planar graphs recently counted

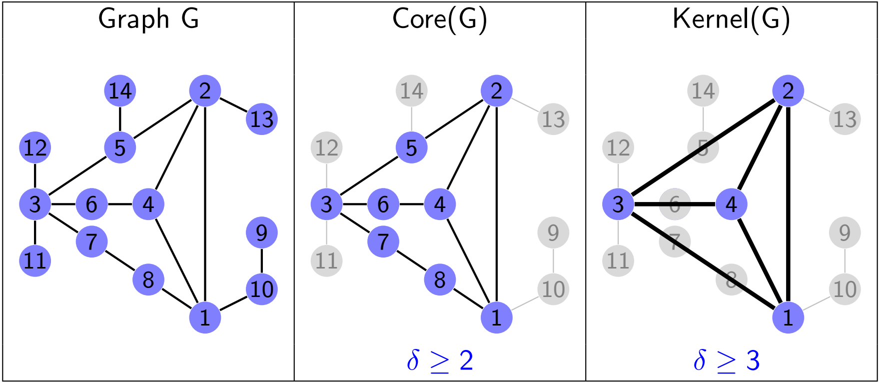
Theorem (Giménez-Noy [2009])

$$g_n \sim c n^{-7/2} \gamma^n n!$$

Goal: extend this and other results to planar $\{2,3\}$-graphs

Tools

Core and Kernel of a graph



Tools

Remark

The kernel of a simple graph might be a multigraph

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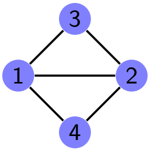

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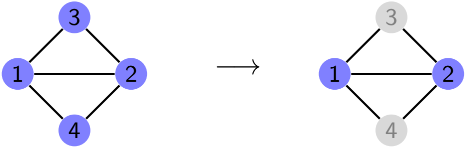
The kernel of a simple graph might be a multigraph



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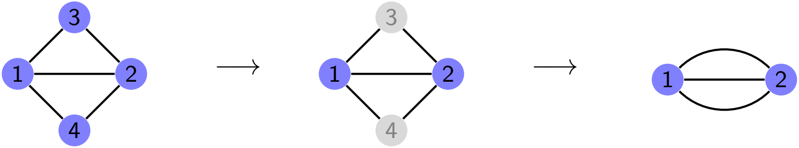

Tools

Labelled structures use Exponential Generating Functions

Exponential Generating Functions

g_n = number of objects of size n of a combinatorial class G

$$G(x) = \frac{g_0}{0!} + \frac{g_1}{1!}x + \frac{g_2}{2!}x^2 + \ldots + \frac{g_k}{k!}x^k + \ldots$$

Starting point

We know generating function of connected planar graphs, $C(x)$

- No explicit expression for $C(x)$
- Solution of an system of equations
- Main properties are known

Disconnected planar graphs

- $F(x)$ EGF of a combinatorial class F
- $e^{F(x)}$ EGF of sets over F
- $G(x) = e^{C(x)}$ EGF of planar graphs
- $C(x) = \log(G(x))$ EGF of connected planar graphs

Connected planar graphs

The generating function $C(x)$ is:

$$C(x) = \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{4}{3!}x^3 + \frac{38}{4!}x^4 + \frac{727}{5!}x^5 + \dots$$

Connected planar graphs

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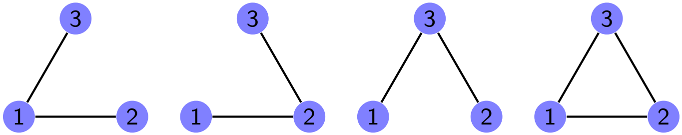

Connected planar graphs

Multivariate EGF that counts both vertices and edges

$$C(x,y) = \frac{1}{1!}x + \frac{1}{2!}x^2y + \frac{3}{3!}x^3y^2 + \frac{1}{3!}x^3y^3 + \dots$$

Labels

- Vertices are labelled, edges are not
- Denominator is $n!$, where n is the number of vertices

Asymptotic properties

Asymptotic properties of random planar graphs

- Number of planar graphs with n vertices
- Number of connected components.
- Size of the biggest connected component
- Expected number of edges
- Number of vertices of degree $k \geq 1$

Asymptotic properties

Asymptotic properties of random planar graphs

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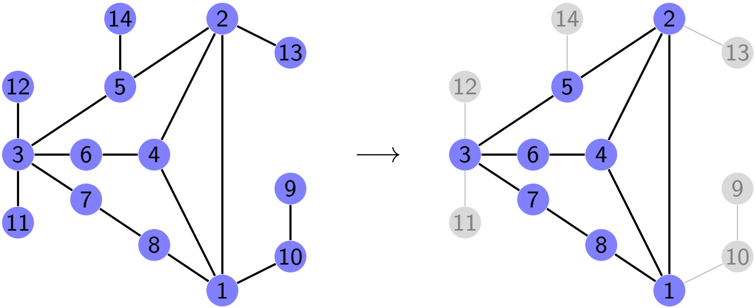
- Number of vertices of degree $k \geq 1$ $d_k n$

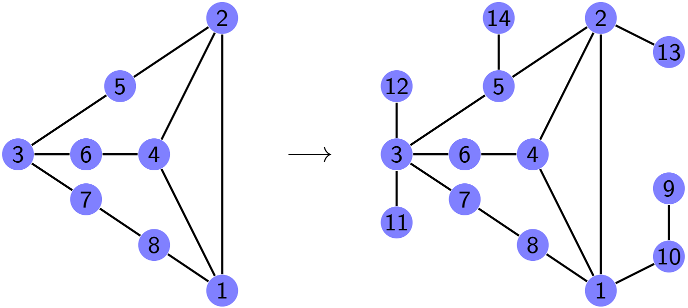
2-graphs

Core of a graph: successively removing vertices of degree 1



2-graphs

Connected planar graph obtained from its core, replacing each vertex by a rooted tree



2-graphs

$H(x)$ generating function of planar 2-graphs

Theorem

$$C(x) = H(T(x)) + U(x)$$

$$T(x) = xe^{T(x)}$$

$$U(x) = T(x) - \frac{T(x)^2}{2}$$

- $T(x)$ generating function of rooted trees
- $U(x)$ generating function of unrooted trees

2-graphs

Generating function of planar 2-graphs can be obtained from connected planar graphs

Lemma

$$T^{-1}(z) = ze^{-z}$$

Theorem

$$H(x) = C(xe^{-x}) - x + \frac{x^2}{2}$$

Using Maple:

$$H(x) = \frac{1}{3!}x^3 + \frac{10}{4!}x^4 + \frac{252}{5!}x^5 + \dots$$

2-graphs

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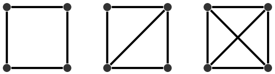

3 6 1

Number of graphs

Exponential growth depends on the radius of convergence

Theorem

$$f(x) = \sum_{n \geq 0} f_n x^n$$

$$\rho = \frac{1}{\limsup \sqrt[n]{f_n}}$$

$f_n \sim \theta(n)\rho^{-n}$, θ subexponential, ρ radius of convergence of $f(x)$

$\rho = 0.03672\ldots$ is the radius of convergence of $C(x)$

$$\rho^{-1} = 27.2268\ldots = \gamma$$

σ radius of convergence of $H(x)$

Number of graphs

$$C(x) = H(T(x)) + U(x)$$

Remark

The radius of convergence of $U(x)$ is bigger than the one of $C(x)$

- $C(x)$ converges if $|x| < \rho$
- $H(y)$ converges if $|y| < T(\rho) = 0.03815\ldots = \sigma$
- $\sigma^{-1} = 26.2075\ldots < 27.2268\ldots = \rho^{-1}$
- Less planar 2-graphs than planar graphs

Expected number of edges

Consider generating function $C(x, y)$, where y counts edges

Theorem

- Function $\rho(y)$ such that $(\rho(y), y)$ singularity of $C(x, y)$
- Quasi-powers theorem: $\mu_C = -\frac{\rho'(1)}{\rho(1)}$ is the proportion between the parameters counted by x and y

Connected planar graphs: $\mu_C = 2.2133\ldots$

Expected number of edges

Equations are adapted so that they consider edges

Theorem

$$C(x,y) = H(T(x,y),y) + U(x,y)$$

$$H(x,y) = C(xe^{-xy},y) - x + \frac{x^2y}{2}$$

Expected number of edges

- Function σ such that $(\sigma(y), y)$ singularity of $H(x, y)$
- Solve the equation $H(\sigma(y), y) = C(\rho(y), y)$
- After replacing H we get $\sigma(y)e^{-\sigma(y)y} = \rho(y)$
- $\mu_H = -\frac{\sigma'(1)}{\sigma(1)} = 2.2613\ldots$

Remark

$$2.2613\ldots = \mu_H > \mu_C = 2.2133\ldots$$

After removing vertices of degree 1 the average degree is higher

Size of the core

Variable u counts vertices in the core

$$C(x,u) = \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{3}{3!}x^3 + \frac{1}{3!}x^3u^3 + \frac{16}{4!}x^4 + \frac{12}{4!}x^4u^3 + \frac{10}{4!}x^4u^4 \dots$$

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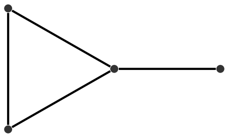

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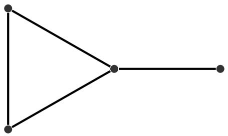

Theorem

$$C(x,u) = H(uT(x)) + U(x)$$

Size of the core

Same technique as in the number of edges

- Function τ such that $C(\tau(u), u)$ is a singularity
- Compute $\tau(1)$ and $\tau'(1)$
- Expected size of the core: $\bar{\mu}n$, where $\bar{\mu} = -\frac{\tau'(1)}{\tau(1)}$
- $\bar{\mu} = 0.9618\ldots$

3-graphs

Different idea for 3-graphs and 2-graphs.

- Cannot use the 3-core
- Use of the Kernel
- Problem: multigraphs

Theorem

$$K(x,y) = C(A(x,y), B(x,y)) + E(x,y)$$

$$A(x,y) = xe^{(x^2y^3-2xy)/(2+2xy)}$$

$$B(x,y) = (y+1)e^{-xy^2/(1+xy)} - 1$$

$$E(x,y) = -x + \frac{x^2y}{2+2xy} - \ln\sqrt{1+xy} + \frac{xy}{2} - \frac{(xy)^2}{4}$$

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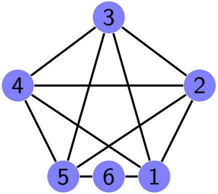

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Comparison

Parameter	Planar Graphs	2-graphs	3-graphs
Growth	$c_1 n^{-7/2}\gamma_1^n n!$ $\gamma_1 = 27.2\ldots$	$c_2 n^{-7/2}\gamma_2^n n!$ $\gamma_2 = 26.2\ldots$	$c_3 n^{-7/2}\gamma_3^n n!$ $\gamma_3 = 21.3\ldots$
Number of edges	$\mu_1 n$ $\mu_1 = 2.21\ldots$	$\mu_2 n$ $\mu_2 = 2.26\ldots$	$\mu_3 n$ $\mu_3 = 2.32\ldots$
Core/ Kernel size	$\mu_1 n$ $\bar{\mu}_1 = 0.96\ldots$	$\mu_2 n$ $\bar{\mu}_2 = 0.82\ldots$	-
Degree distri-bution	$d_k n$	$d'_k n$	$d''_k n$

Thank you for your attention