Discrete Serrin's Problem

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The original Serrin's Problem

Given $\Omega \subset \mathbb{R}^n$, with smooth boundary $\delta(\Omega)$, if u is the unique solution of

$$-\Delta(u) = 1 \quad \text{on } \Omega$$

$$u = 0 \quad \text{on } \delta(\Omega)$$

then $\dfrac{\partial u}{\partial \mathrm{n}}$ is constant iff Ω is a ball and $u(x) = \dfrac{1}{2n}(R^2 - |x|^2)$; that is, u is radial.

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- Moving planes
- Minimum principle and Green Identities

Discrete Serrin's Problem

Given $\Gamma = \big(F \cup \delta(F), c\big)$ a network with boundary if u is the unique solution of

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- Minimum principle and Green Identities
- Existence of equilibrium measure

Network Topology

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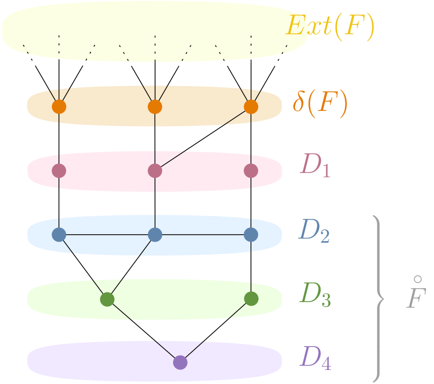
- **Given** $F \subset V$ consider the sets



$r(F) = \max_{x \in F} \{d(x, \delta(F)\}$

Operators

- **Combinatorial Laplacian** $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y)\big(u(x) - u(y)\big) = k(x)u(x) - \sum_{y \in V} c(x,y)u(y)$$

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- **Matrix version**

$$\begin{pmatrix} k(x_1) & -c(x_1,x_2) & \cdots & -c(x_1,x_n) \\ -c(x_1,x_2) & k(x_2) & \cdots & -c(x_2,x_n) \\ \vdots & \vdots & \ddots & \vdots \\ -c(x_1,x_n) & -c(x_2,x_n) & \cdots & k(x_n) \end{pmatrix} = D - A$$

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- **Normal derivative:** $u \in \mathcal{C}(V)$ and F connected proper set

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Gauss Theorem: $$\sum_{x\in F} \mathcal{L}(u)(x) = -\sum_{x\in\delta(F)} \frac{\partial u}{\partial \mathsf{n}}(x)$$

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- If $u \in \mathcal{C}(V)$ is superharmonic on F, then

$$\min_{x \in \delta(F)} \{u(x)\} \leq \min_{x \in F} \{u(x)\}$$

  The equality holds iff $u = a\chi_{\bar{F}}$

Generalized minimum principles

- If $u \in \mathcal{C}(\bar{F})$ is superharmonic on F, then for any $i = 1, \ldots, r(F) - 1$

$$\min_{x \in \delta(F)} \{u(x)\} \leq \min_{x \in D_i} \{u(x)\} \leq \min_{x \in D_{i+1}} \{u(x)\}$$

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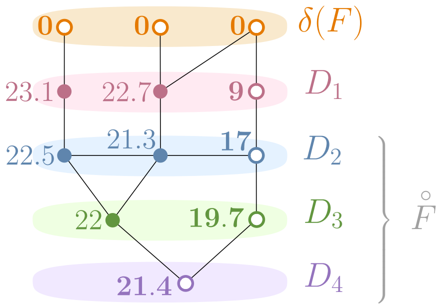

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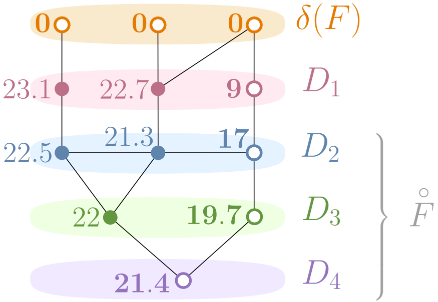

Level sets

- Given $u \in \mathcal{C}^+(F)$ we denote $0 = u_0 < u_1 < \cdots < u_s$
  - **Level set** $U_i = \{x \in F : u(x) = u_i\}$

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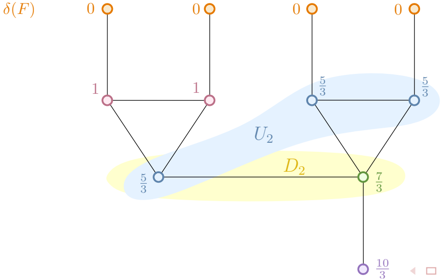

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- If $u \in \mathcal{C}^+(F)$ is a strictly superharmonic function on F satisfying $U_j = D_j$ for all $j = 0, \ldots, i$, then $U_{i+1} \subset D_{i+1}$

Radial Functions

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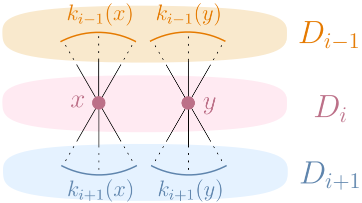

Radial Functions

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- If $u \in \mathcal{C}^+(F)$ is a <u>radial function</u>, then for any $x \in D_i$

$$\mathcal{L}u(x) = \underbrace{k_{i+1}(x)}_{\sum_{y \in D_{i+1}} c(x,y)} (u_i - u_{i+1}) + \underbrace{k_{i-1}(x)}_{\sum_{y \in D_{i-1}} c(x,y)} (u_i - u_{i-1}) > 0$$

- Moreover, for any $x \in D_0$,

$$\frac{\partial u}{\partial \mathsf{n}}(x) = -k_1(x) u_1 < 0$$

Serrin's problem

Given $\Gamma = \big(F \cup \delta(F), c\big)$ a network with boundary if ν is the equilibrium measure of F

$$\mathcal{L}(\nu) = 1 \text{ on } F, \quad \nu = 0 \text{ on } \delta(F)$$

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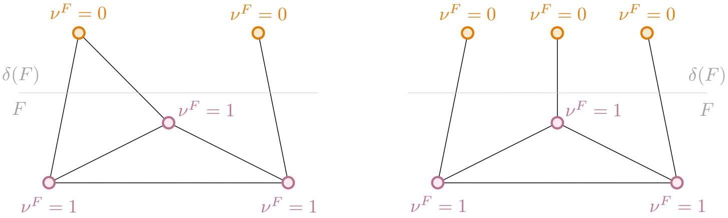
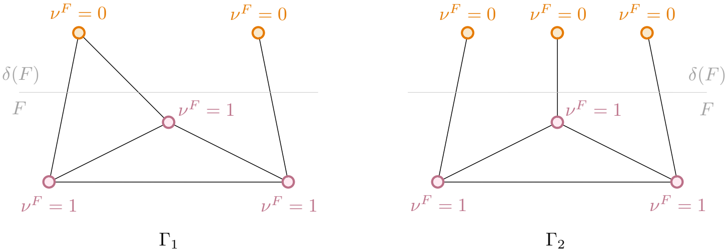
- Has Γ ball-like structure?
- Is ν radial?
- If ν satisfies Serrin's condition, then $C = -\dfrac{|F|}{|\delta(F)|}$

Separated boundary



Γ_1 Γ_2

Separated boundary



- For any $x \in \delta(F)$ there $\exists!$ $\hat{x} \in D_1$ such that $c(x, \hat{x}) > 0$

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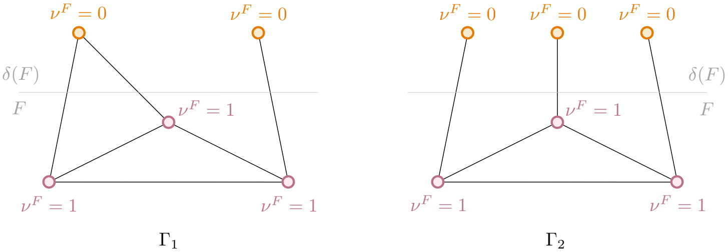

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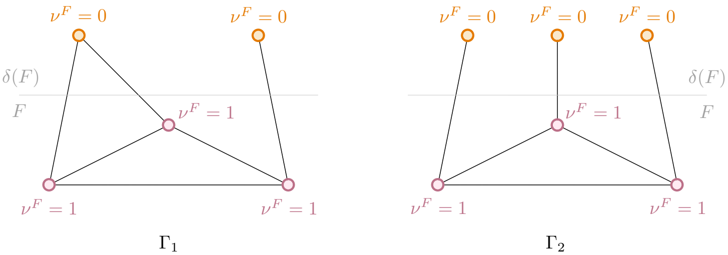

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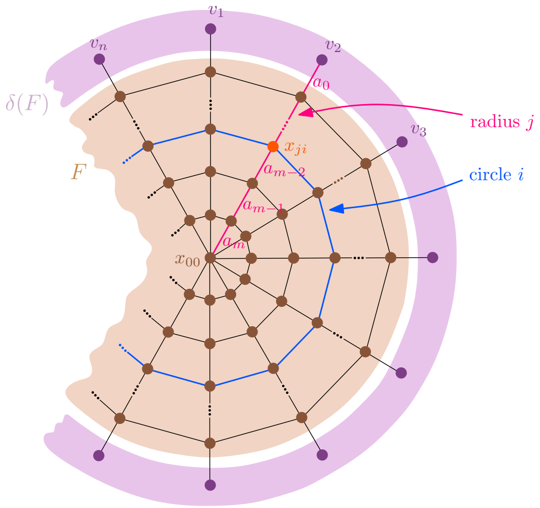
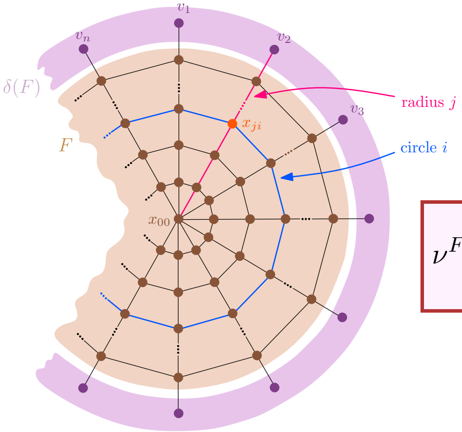
- We suppose that $|F| \geq 2$
- If ν satisfies Serrin's condition, then $U_1 = D_1$ iff $c(x, \hat{x})$ is constant. Therefore, $U_2 \subset D_2$

Spider network with radial conductances



m circles
n radius

Spider network with radial conductances



$$\nu^F(x_{js}) = \frac{1}{n}\sum_{i=0}^{m-s}\frac{n(m-i)+1}{a_i}$$

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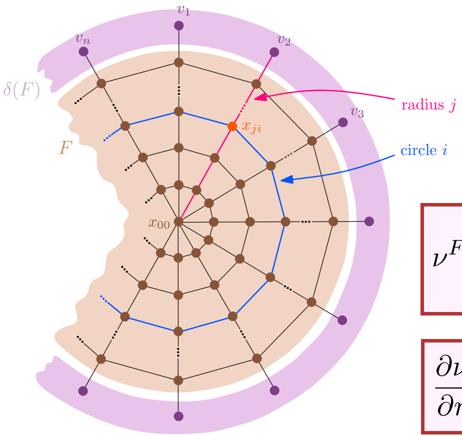
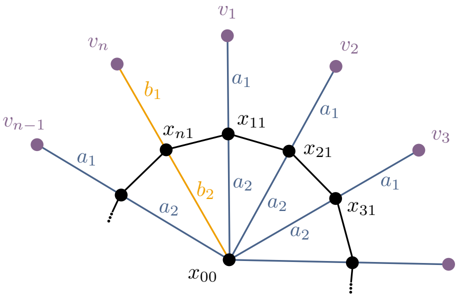


$$\nu^F(x_{js}) = \frac{1}{n}\sum_{i=0}^{m-s} \frac{n(m-i)+1}{a_i}$$

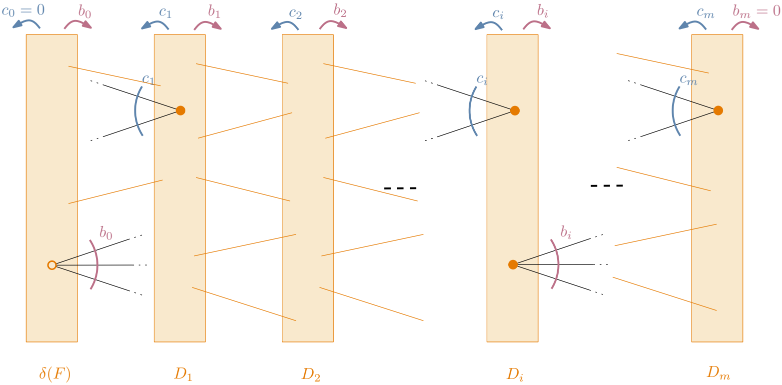
$$\frac{\partial \nu^F}{\partial \eta_F} = -\frac{(nm+1)}{n}$$

Spider network



- Serrin's condition holds iff $b_1 = a_1$ and $b_2 = a_2$

Regular Layered Networks



- For any $x \in D_i$, $k_{i-1}(x) = c_i$ and $k_{i+1}(x) = b_i$, $i = 1, \ldots, m$

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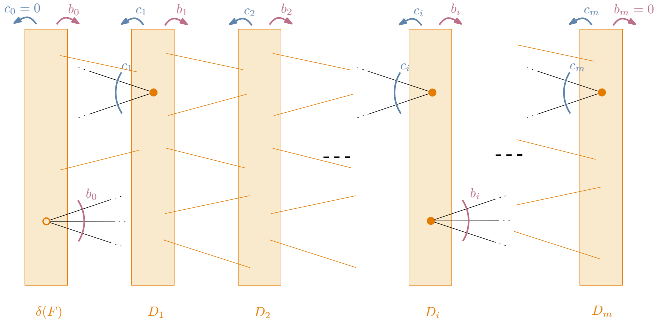

- $$\nu^F(x) = \sum_{j=1}^{s} \frac{1}{b_{j-1}} \sum_{k=j}^{m} \left(\prod_{\ell=j-1}^{k-1} \frac{b_\ell}{c_{\ell+1}} \right) \quad \text{for all} \quad x \in D_s$$

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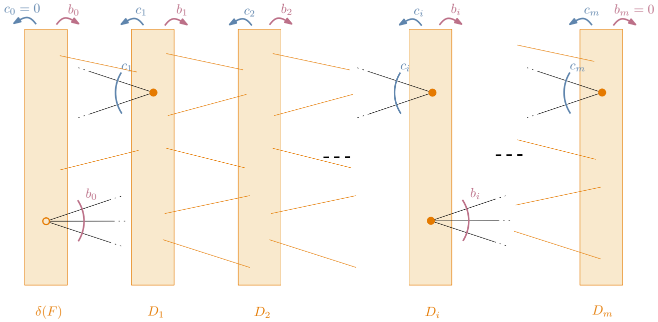

$$\frac{\partial \nu^F}{\partial \mathsf{n}}(x) = -\sum_{i=1}^{m}\left(\prod_{\ell=0}^{i-1}\frac{b_\ell}{c_{\ell+1}}\right) \quad \text{for all} \quad x \in D_0$$

Characterization

Let Γ be a network such that for all $i = 1, \dots, m-1$,

$$k_{i+1}(x) + k_{i-1}(x) = d_i \quad \text{for all} \quad x \in D_i$$

$$U_i = D_i$$

$$m = s$$

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- Then, ν satisfies Serrin's condition iff Γ is a layered regular graph