Discrete Serrin’s Problem

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The original Serrin’s Problem

Given $\Omega \subset \mathbb{R}^n$, with smooth boundary $\delta(\Omega)$, if $u$ is the unique solution of

$$-\Delta(u) = 1 \quad \text{on } \Omega$$
$$u = 0 \quad \text{on } \delta(\Omega)$$

then $\frac{\partial u}{\partial n}$ is constant iff $\Omega$ is a ball and $u(x) = \frac{1}{2n} (R^2 - |x|^2)$; that is, $u$ is radial.
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- Moving planes
- Minimum principle and Green Identities
Discrete Serrin’s Problem

Given $\Gamma = (F \cup \delta(F), c)$ a network with boundary if $u$ is the unique solution of

$$\mathcal{L}(u) = 1 \quad \text{on } F$$

$$u = 0 \quad \text{on } \delta(F)$$

then if $\frac{\partial u}{\partial n}$ is constant, what can we say about $\Gamma$ and $u$?
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Minimum principle and Green Identities
Discrete Serrin’s Problem

Given \( \Gamma = (F \cup \delta(F), c) \) a network with boundary if \( u \) is the unique solution of

\[
\mathcal{L}(u) = 1 \quad \text{on } F \\
u = 0 \quad \text{on } \delta(F)
\]

then if \( \frac{\partial u}{\partial n} \) is constant, what can we say about \( \Gamma \) and \( u \)?

- Minimum principle and Green Identities
- Existence of equilibrium measure
Network Topology

- Network $\Gamma = (V, E, c)$
Network Topology

- **Network** $\Gamma = (V, E, c)$

- **Given** $F \subset V$ consider the sets

\[
r(F) = \max_{x \in F} \{d(x, \delta(F))\}
\]
Operators

- **Combinatorial Laplacian** $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$

\[
\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y)(u(x) - u(y)) = k(x)u(x) - \sum_{y \in V} c(x, y)u(y)
\]
Operators

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\[
\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) (u(x) - u(y)) = k(x) u(x) - \sum_{y \in V} c(x, y) u(y)
\]

- **Matrix version**

\[
\begin{pmatrix}
k(x_1) & -c(x_1, x_2) & \cdots & -c(x_1, x_n) \\
-c(x_1, x_2) & k(x_2) & \cdots & -c(x_2, x_n) \\
\vdots & \vdots & \ddots & \vdots \\
-c(x_1, x_n) & -c(x_2, x_n) & \cdots & k(x_n)
\end{pmatrix} = D - A
\]
Operators

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  \]

- **Normal derivative**: \( u \in \mathcal{C}(V) \) and \( F \) connected proper set

  \[
  \frac{\partial u}{\partial n}(x) = \sum_{y \in F} c(x, y)(u(x) - u(y)), \quad \text{for any } x \in \partial(F)
  \]
Operators

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\frac{\partial u}{\partial n}(x) = \sum_{y \in F} c(x, y)(u(x) - u(y)), \text{ for any } x \in \delta(F)
\]

**Gauss Theorem**:

\[
\sum_{x \in F} \mathcal{L}(u)(x) = -\sum_{x \in \delta(F)} \frac{\partial u}{\partial n}(x)
\]
Minimum principle

- A function $u \in \mathcal{C}(V)$ is called
  - Superharmonic if $\mathcal{L}(u) \geq 0$
Minimum principle

- A function $u \in \mathcal{C}(V)$ is called

  - **Superharmonic** if $\mathcal{L}(u) \geq 0$

  - **Strictly Superharmonic** if $\mathcal{L}(u) > 0$
Minimum principle

- A function $u \in C(V)$ is called &nbsp;Superharmonic if $\mathcal{L}(u) \geq 0$
- Strictly Superharmonic if $\mathcal{L}(u) > 0$

If $u \in C(V)$ is superharmonic on $F$, then

$$\min_{x \in \delta(F)} \{u(x)\} \leq \min_{x \in F} \{u(x)\}$$

The equality holds iff $u = a \chi_{\overline{F}}$. 
Generalized minimum principles

If $u \in C(\bar{F})$ is superharmonic on $F$, then for any $i = 1, \ldots, r(F) - 1$

$$\min_{x \in \delta(F)} \{u(x)\} \leq \min_{x \in D_i} \{u(x)\} \leq \min_{x \in D_{i+1}} \{u(x)\}$$
Generalized minimum principles

If \( u \in C(\overline{F}) \) is superharmonic on \( F \), then for any \( i = 1, \ldots, r(F) - 1 \)

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\]
Generalized minimum principles

If $u \in \mathcal{C}(\bar{F})$ is superharmonic on $F$, then for any $i = 1, \ldots, r(F) - 1$

$$\min_{x \in \delta(F')} \{u(x)\} \leq \min_{x \in D_i} \{u(x)\} \leq \min_{x \in D_{i+1}} \{u(x)\}$$

If $u \in \mathcal{C}^+(F)$ is a strictly superharmonic function on $F$, then for any $x \in F$ there exists $y \in \bar{F}$ such that $c(x, y) > 0$ and $u(y) < u(x)$
If $u \in C^+(F)$ is a strictly superharmonic function on $F$, then for any $x \in F$ there exists $y \in \bar{F}$ such that $c(x, y) > 0$ and $u(y) < u(x)$.
Level sets

Given $u \in C^+(F)$ we denote $0 = u_0 < u_1 < \cdots < u_s$

Level set $U_i = \{x \in F : u(x) = u_i\}$
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If $u \in C^+(F)$ is a strictly superharmonic function on $F$, then $U_0 = D_0$ and $U_i \subset \bigcup_{j=1}^{i} D_i$, for any $i = 1, \ldots, s$
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If \( u \in C^+(F) \) is a strictly superharmonic function on \( F \), then
\[
U_0 = D_0 \quad \text{and} \quad U_i \subset \bigcup_{j=1}^i D_i, \text{ for any } i = 1, \ldots, s
\]

If \( u \in C^+(F) \) is a strictly superharmonic function on \( F \) satisfying \( U_j = D_j \) for all \( j = 0, \ldots, i \), then \( U_{i+1} \subset D_{i+1} \)
An strictly superharmonic function $u \in C^+(F)$ is called \textbf{radial} if $U_i = D_i$, for any $i = 0, \ldots, s$.
Radial Functions

An strictly superharmonic function $u \in \mathcal{C}^+(F)$ is called radial if $U_i = D_i$, for any $i = 0, \ldots, s \implies s = r(F)$
Radial Functions

An strictly superharmonic function $u \in C^+(F)$ is called <u>radial</u> if $U_i = D_i$, for any $i = 0, \ldots, s \implies s = r(F)$

If $u \in C^+(F)$ is a <u>radial function</u>, then for any $x \in D_i$

$$\mathcal{L}u(x) = k_{i+1}(x)(u_i - u_{i+1}) + k_{i-1}(x)(u_i - u_{i-1}) > 0,$$
Radial Functions

- An strictly superharmonic function \( u \in C^+(F) \) is called **radial** if \( U_i = D_i \), for any \( i = 0, \ldots, s \implies s = r(F) \)

If \( u \in C^+(F) \) is a radial function, then for any \( x \in D_i \)

\[
\mathcal{L}u(x) = \sum_{y \in D_{i+1}} c(x,y) (u_i - u_{i+1}) + \sum_{y \in D_{i-1}} c(x,y) (u_i - u_{i-1}) > 0
\]

![Diagram of radial functions and their domains](attachment:image.png)
Radial Functions

An strictly superharmonic function $u \in \mathcal{C}^+(F)$ is called radial if $U_i = D_i$, for any $i = 0, \ldots, s \implies s = r(F)$

If $u \in \mathcal{C}^+(F)$ is a radial function, then for any $x \in D_i$

$$\mathcal{L}u(x) = \sum_{y \in D_{i+1}} c(x,y) (u_i - u_{i+1}) + \sum_{y \in D_{i-1}} c(x,y) (u_i - u_{i-1}) > 0$$

Moreover, for any $x \in D_0$,

$$\frac{\partial u}{\partial n}(x) = -k_1(x)u_1 < 0$$
Serrin’s problem

Given $\Gamma = (F \cup \delta(F), c)$ a network with boundary if $\nu$ is the equilibrium measure of $F$

$$\mathcal{L}(\nu) = 1 \text{ on } F, \quad \nu = 0 \text{ on } \delta(F)$$

then if $\frac{\partial \nu}{\partial n} = C$, what can we say about $\Gamma$ and $\nu$?
Serrin’s problem

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▷ Has \( \Gamma \) ball-like structure?
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▶ Has $\Gamma$ ball-like structure?

▶ Is $\nu$ radial?
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then if \( \frac{\partial \nu}{\partial n} = C \), what can we say about $\Gamma$ and $\nu$?

▶ Has $\Gamma$ ball-like structure?

▶ Is $\nu$ radial?

▶ If $\nu$ satisfies Serrin’s condition, then $C = -\frac{|F|}{|\delta(F)|}$
Separated boundary

\[ \nu^F = 0 \quad \nu^F = 0 \]

\[ \Gamma_1 \]

\[ \delta(F) \]

\[ F \]

\[ \nu^F = 1 \quad \nu^F = 1 \]

\[ \nu^F = 1 \quad \nu^F = 1 \]

\[ \Gamma_2 \]

\[ \delta(F) \]

\[ F \]
For any \( x \in \delta(F) \) there \( \exists! \) \( \hat{x} \in D_1 \) such that \( c(x, \hat{x}) > 0 \)
Separated boundary

For any $x \in \delta(F)$ there $\exists! \hat{x} \in D_1$ such that $c(x, \hat{x}) > 0$

We suppose that $|F| \geq 2$
Separated boundary

- For any $x \in \delta(F)$ there $\exists! \, \hat{x} \in D_1$ such that $c(x, \hat{x}) > 0$

- We suppose that $|F| \geq 2$

- If $\nu$ satisfies Serrin’s condition, then $U_1 = D_1$ iff $c(x, \hat{x})$ is constant. Therefore, $U_2 \subset D_2$
Spider network with radial conductances

\[ F, \quad \delta(F), \quad x_{ji}, \quad a_0, \quad a_{m-1}, \quad a_{m-2}, \quad x_{00}, \quad v_n, \quad v_1, \quad v_2, \quad v_3, \quad \text{circle } i, \quad \text{radius } j, \quad m \text{ circles, } \quad n \text{ radius} \]
Spider network with radial conductances

\[ \nu^F(x_{js}) = \frac{1}{n} \sum_{i=0}^{m-s} \frac{n(m-i) + 1}{a_i} \]
Spider network with radial conductances

$$\nu^F(x_{js}) = \frac{1}{n} \sum_{i=0}^{m-s} \frac{n(m-i)+1}{a_i}$$

$$\frac{\partial \nu^F}{\partial \eta_F} = -\frac{(nm+1)}{n}$$
Serrin’s condition holds iff $b_1 = a_1$ and $b_2 = a_2$.
Regular Layered Networks

For any \( x \in D_i \), \( k_{i-1}(x) = c_i \) and \( k_{i+1}(x) = b_i, \ i = 1, \ldots, m \)
Regular Layered Networks

\[ \nu^F(x) = \sum_{j=1}^{s} \frac{1}{b_{j-1}} \sum_{k=j}^{m} \left( \prod_{\ell=j-1}^{k-1} \frac{b_{\ell}}{c_{\ell+1}} \right) \] for all \( x \in D_s \)
Discrete Serrin’s Problem
Network with boundary
Ball-like structure

Regular Layered Networks

\[
\frac{\partial \nu^F}{\partial n}(x) = -\sum_{i=1}^{m} \left( \prod_{\ell=0}^{i-1} \frac{b_\ell}{c_{\ell+1}} \right) \quad \text{for all } x \in D_0
\]
Characterization

Let $\Gamma$ be a network such that for all $i = 1, \ldots, m - 1$,

$$k_{i+1}(x) + k_{i-1}(x) = d_i \quad \text{for all} \quad x \in D_i$$

$$U_i = D_i$$

$$m = s$$
Characterization

Let $\Gamma$ be a network such that for all $i = 1, \ldots, m - 1$,

$$k_{i+1}(x) + k_{i-1}(x) = d_i \quad \text{for all } x \in D_i$$

$$U_i = D_i$$

$$m = s$$

Then, $\nu$ satisfies Serrin’s condition iff $\Gamma$ is a layered regular graph.