Radical parametrization of algebraic curves and surfaces

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Parametrization of algebraic objects

Implicit representation: $f(X) = 0$

vs.

Parametric representation: $X = g(T)$

Uses of parametric representations:

- intersecting two varieties
- plotting
- computing line or surface integrals
- dealing with the velocity or the acceleration of a particle
- performing geometric transformations, such as rotations, translations, scalings, projections...

Rational parametrization

What type of functions do we want to use for parametrization?

Example: the real unit circle, $x^2 + y^2 = 1$, is parametrized by

$$(\cos(t), \sin(t)), \qquad t \in [0, 2\pi]$$

and also by

$$\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right), \qquad t \in \mathbb{R} \qquad (\textit{almost}\ldots)$$

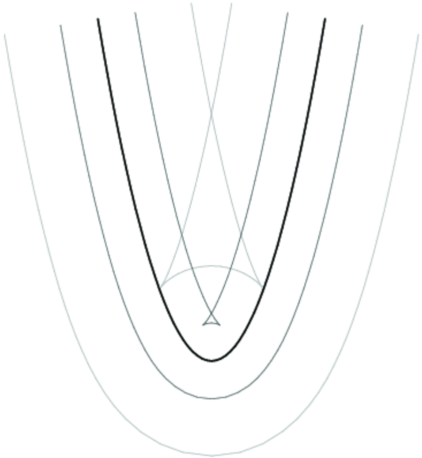
When working with algebraic objects, it is customary to work with rational parametrizations.

Rational parametrization of curves

An algebraic curve can be rationally parametrized iff it has genus 0.

Example: geometric constructions in CAGD

Some very common geometric constructions, like offsets and conchoids of hypersurfaces, do not preserve rationality: they involve the norm of the normal vector, which in general needs a $\sqrt{\ }$.



Radical parametrization of curves

Examples

All elliptic curves (genus 1) can be expressed as $y^2 = x^3 + ax + b$.
This can be parametrized as $x = t$, $y = \sqrt{t^3 + at + b}$.

Similarly for all hyperelliptic curves (genus ≥ 2), $y^2 = P(x)$.

Non-examples

If $F(x, y) = 0$ with degree n in the variable y, we can set $x = t$,
$y =$ (solution of a polynomial of degree n in t).

The problem

Given an algebraic $\mathbb{C}$urve, decide whether it admits a radical parametrization, and in the affirmative case compute one.

What did you say a parametrization is?

Definition of parametrization

Given an implicit curve $F_1(x_1, \ldots, x_n) = \cdots = F_m(x_1, \ldots, x_n)$, a parametrization is a n-uple of "functions" $x_1(t), \ldots, x_n(t)$ such that all substitutions in the F's are zero. Note that this is a formal substitution in some function class! No function evaluation is considered here.

Radical parametrizations

The class of algebraic "functions" that we consider for radical parametrization are elements of some algebraic extension of the rational function field $k(t)$:

$$x_1(t), \ldots, x_n(t) \in k(t)(\alpha) \subset k(t)$$

For example, $t + \sqrt{t} \in k(\alpha)$ where $\alpha^2 - t = 0$. (What does $\sqrt{t}$ mean!)

Gonality

In one variable, the degree d of a polynomial $p(x)$ gives us information on whether we can express its roots by radicals: always when $d \leq 4$, almost never when $d > 4$. A similar concept exists in more variables:

Definition

The gonality of a curve C is the minimum $n \in \mathbb{N}$ such that there exists an $n : 1$ map from C to $\mathbb{P}^1$ (or, if you prefer, the curve has a g^1_n).

- gonality 2 = hyperellipticity
- gonality 3 = trigonality
- etc.

Gonality and parametrizability by radicals

Given a $n : 1$ map, with $n \leq 4$, to express its inverse we need to solve polynomial(s) of degree n in one variable. Thus:

Every curve of gonality $g \leq 4$ can be parametrized by radicals.

Gonality and genus

We are more familiar with the notion of genus of an algebraic curve.

Zariski (1926)

The general complex projective curve of genus $g > 6$ is not parametrizable by radicals.

The concepts of genus and gonality are related:

Relation between gonality and genus

Every curve of genus g has a linear system of dimension 1 and degree $\lceil \frac{g}{2} + 1 \rceil$.

$\Rightarrow$ the gonality of a curve of genus g is $\leq \lceil \frac{g}{2} + 1 \rceil$.

- genus 1, 2 $\Rightarrow$ gonality ≤ 2
- genus 3, 4 $\Rightarrow$ gonality ≤ 3
- ...

The trigonal case: algebraic geometry results

All curves of genus 3 and 4 are trigonal or hyperelliptic.

Theorem (Enriques 1919, Babbage 1939)

Let C be a canonical curve (the image of the canonical embedding for non-hyperelliptic curves). Let Q be the intersection of the quadrics containing it. Then $C = Q$ except when C is trigonal or a plane quintic.

Theorem (Griffiths & Harris 1978)

Any canonical curve C satisfies exactly one of these:

- $C = Q$;
- C is trigonal, and Q is the rational normal scroll swept out by the trisecants;
- C is a plane quintic, and Q is the Veronese surface in $\mathbb{P}^5$, swept out by the conic curves through five coplanar points of the curve.

The Lie algebra method

Problem

Let M be a known variety (a model) and X an arbitrary variety. We want to recognize constructively whether $X \cong M$.

Idea

1. Precompute or load the Lie algebra associated to M, $L(M)$.
2. Compute $L(X)$.
3. If we can constructively recognize $L(X) \cong L(M)$ or parts of them, sometimes we can go back with that and get an isomorphism $X \longleftrightarrow M$ (via Lie algebra representations).

Disclaimer: a Lie algebra is a manageable object from a computational point of view (imagine linear spaces of $(n+1) \times (n+1)$ matrices).

The Lie algebra method in our situation

We denote the rational normal scroll with parameters m, n as $S_{m,n}$, the Veronese surface as V, and curves as C.

Recognition

If we write $LSA(X)$ for the Levi subalgebra of the Lie algebra $L(X)$:

1. $LSA(S_{m,n}) \cong \mathfrak{sl}_2$ (dimension 3) if $m \neq n$;
2. $LSA(S_{m,m}) \cong \mathfrak{sl}_2 + \mathfrak{sl}_2$ (dimension 6);
3. $LSA(V) \cong \mathfrak{sl}_3$ (dimension 8);
4. $LSA(C)$ is trivial.

Constructive recognition

In order to build an isomorphism between $LSA(X)$ and the LSA of our model, in the end we just need to compute a certain basis of eigenvectors of the underlying k^{n+1}.

The tetragonal case and beyond

All curves of genus 5 and 6 are tetragonal or belong to the previous cases. More in general:

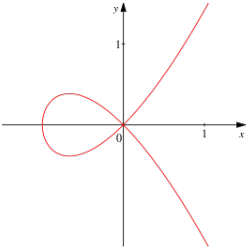
Tetragonal case

- Harrison 2012: solved for genus 5 and 6. Uses free minimal resolutions.
- Schicho, Schreyer & Weimann 2013:
  - A deterministic algorithm that calculates the gonality of a given curve and a map that realizes the gonality. Uses syzygyes $\Rightarrow$ limited practical usability.
  - An algorithm for the case of gonality ≤ 4 in characteristic $\neq 2, 3$.

Still, parametrization by radicals using gonality maps is limited to the extent that we cannot solve, in general, polynomials of degree > 4.

Another approach: adjoint curves (I)

One can parametrize (rationally) a singular cubic curve considering a 1-parameter family of lines.



Using the same idea we can parametrize radically all:

- Curves of degree ≤ 5.
- Singular curves of degree 6.
- Curves of degree d with a singular point of multiplicity $d - r$, $r \leq 4$.

Another approach: adjoint curves (II)

One can parametrize (rationally) a rational curve considering a 1-dimensional family of adjoint curves. This technique can be used for radical parametrizations of certain curves (Sendra & Sevilla 2011):

Theorem

Let C be a curve of degree d and low genus. Then it can be parametrized by radicals:

- genus $g \leq 3 \to$ using adjoint curves of degree $d - 2$;
- genus $2 \leq g \leq 4 \to$ using adjoint curves of degree $d - 3$.

The method is weaker than results previously shown: it produces a $g : 1$ map for a curve of genus g (suboptimal for $g = 4, 5, 6$).

Radical parametrization of algebraic surfaces

Given an irreducible implicit algebraic surface $F(x, y, z) = 0$, can we find a rational parametric representation $x(s, t), y(s, t), z(s, t)$ of it?

In general, no: the genera must be 0. So, once more we extend the problem to “functions” in s, t expressible by radicals.

Example

$$z^2 - xy = 0 \longleftarrow (s, t, \sqrt{st})$$

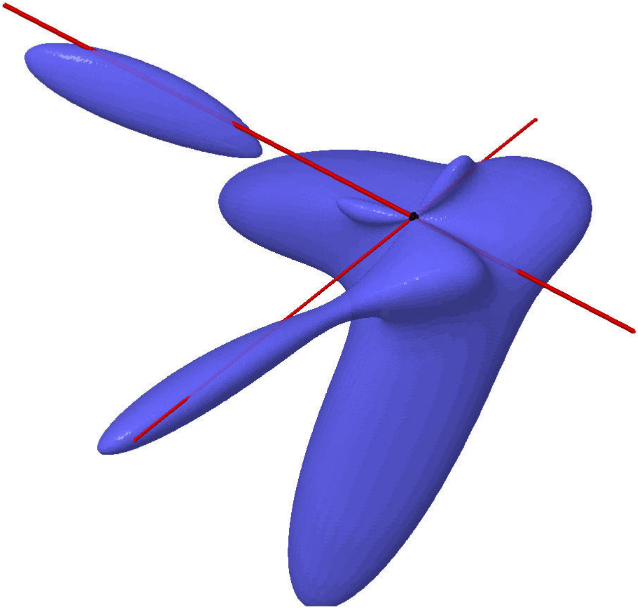
Easy cases

- If the degree in one of the variables is ≤ 4, the surface is radically parametrizable.

Using parametrization by a pencil of lines:

- Every surface of degree 5 is radically parametrizable.
- Every surface of degree 6 with a singular point is radically parametrizable.

Radical parametrization by lines



Radical parametrization of surfaces as curves

We can use radical parametrization of curves to find radical parametrizations of surfaces:

"Regard $F(x, y, z)$ as a polynomial in x, y with coefficients in $k(z)$."

This allows us to parametrize surfaces that have a pencil of curves, provided we can parametrize those curves by radicals.

Excuse me! There is a *caveat*...

If our adjoint method uses "a point on the curve", we need it to have radical coordinates over $k(z)$ (now z is part of the coefficients)!

Curves by adjoints in more detail

For a curve of degree d and genus g:

Case $1 \leq g \leq 3$

1. Compute the adjoints of degree $d - 2$
2. Restrict them to passing by $(d - 3) + g$ simple points on the curve

Case $2 \leq g \leq 4$

1. Compute the adjoints of degree $d - 3$
2. Restrict them to passing by $g - 2$ simple points on the curve

Remember the *caveat*...

Surfaces with a pencil of low-genus curves

Pencil of genus 1 curves

We need two simple radical points. It suffices to have one of these:

- One double point.
- One r-fold point and $r - 2$ simple radical point.

Pencil of genus 2 curves

We need no simple points.

Pencil of genus 3 curves

We need one simple radical point. A general adjoint intersects the curve in 4 simple points (thus radical). Take any one of them.

Pencil of genus 4 curves

We need two simple radical points.

Geometric constructions of low degree are radical

We mentioned in the beginning that some common geometric constructions in CAGD do not preserve rationality.

On the other hand, offsets and conchoids of radical curves and surfaces are always radical. More generally:

Theorem

Let S be a radical irreducible surface, and let $\mathcal{Z}$ be the geometric variety generated from S via a geometric construction of degree at most 4. Then $\mathcal{Z}$ is radical.

Future work

- General techniques for representabilty by radicals (Galois theory)
- Other questions: normality, simplification, working over the reals...
- The real world!

Thank you!