Dimensional curvature identities

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(Joint work with J. Navarro)

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Main references

- **Gilkey, P.**
  - *Curvature and eigenvalues of the Laplacian for elliptic complexes* (1973)
- **Gilkey, P.; Park, J.H.; Sekigawa, K.**
  - *Universal curvature identities* (2011)
- **Gilkey, P.; Park, J.H.; Sekigawa, K.**
  - *Universal curvature identities II* (2012)

Motivation

Let (X, g) be a Riemannian manifold.

Then coefficients of the curvature tensor R satisfy certain identities:

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \quad , \quad \text{Bianchi identity}$$

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0 \quad , \quad \text{Differential Bianchi identity}$$

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These identities have the following properties:

- They are independent of chart choices.
- They are satisfied on any Riemannian manifold, regardless of the dimension.

This first property suggests that there is an intrinsic description of them:

$$R \wedge I = 0 \quad \text{Bianchi identity}$$

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These two identities (and those obtained from it using covariant differentiation) can be proved to be, essentially, the only identities satisfied by the curvature with those two properties.

Are there other identities that only occur on certain dimensions? Yes. For example, if $\dim X = 2$ the Einstein tensor vanishes. That is

$$R_{ij} = \frac{r}{2} g_{ij} \quad \text{i.e.} \quad Ricc = \frac{r}{2} g$$

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Natural tensors in dimension n

Tensors which do not rely on chart choices for their definition are called *natural tensors*.

More concretely, let X be a manifold of dim n. Let *Metrics* and p-*Tensors* be the sheaves of smooth sections of the bundle of Riemannian metrics and the bundle of p-contravariant tensors on X.

These two bundles are *natural*, in the sense that any diffeomorphism $\tau \colon U \to V$ between open sets of X acts, by push-forward τ_*, on both metrics and p-tensors.

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Definition

A p-contravariant **natural tensor** associated to a metric **in dimension** n is a regular morphism of sheaves:

$$T\colon \textit{Metrics} \longrightarrow p\text{-}\textit{Tensors}\ ,$$

such that, for any diffeomorphism $\tau\colon U \to V$ between open sets of X, it satisfies

$$T(\tau_* g) = \tau_* T(g)\ .$$

It is said **homogeneous of weight** $w \in \mathbb{R}_+$ if for every $\lambda \in \mathbb{R}$ we have $T(\lambda g) = \lambda^w T(g)$.

It can be proved that the local coefficients of a natural tensor are certain "universal" (i.e., valid on any chart) smooth functions on the coefficients of the metric, its inverse, the curvature and its covariant derivatives.

As an example, if a natural tensor is polynomial, then it is obtained by means of tensor products, contractions and covariant differentiation of the following three tensors:

$$g\ ,\ g^{-1}\ ,\ R\ .$$

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Let (X, g) be a Riemannian manifold of dimension $n-1$. The manifold $X \perp \mathbb{R}$ with metric $g \otimes \mathrm{d}t^2$ is an Riemannian manifold of dimension of dimension n.

Let $\mathrm{T}^w[n]$ be the natural tensors of dimension n. For any $T \in \mathrm{T}^w[n]$ we may consider

$$r_n(T) : X \to X \perp \mathbb{R} \xrightarrow{T} \mathbb{R}$$

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Definition

We call $r_n : \mathrm{T}^w[n] \to \mathrm{T}^w[n-1]$ the **dimensional reduction** map.

Definition

We have a sequence

$$\cdots \xrightarrow{r_{n+1}} \mathrm{T}^w[n] \xrightarrow{r_n} \mathrm{T}^w[n-1] \xrightarrow{r_{n-1}} \cdots$$

A **universal tensor** is an element of the inverse limit

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Example: The difference between universal tensor and natural tensors is subtle

- The metric g itself is a universal tensor, and for any fixed $\lambda \in \mathbb{R}$ the tensor λg is also universal.
- However, $n \cdot g$ where $n = \dim X$ is not a universal tensor. Nor $(-1)^n g$.
- The Ricci tensor *Ricc* and the scalar curvature r are a universal tensors.
- The volume form $\mathrm{d}x$ is not a universal tensor.

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The theorem

Theorem

Let n be an even integer.
If $w = -(n+p)$, the kernel of the restriction map

$$\mathrm{T}^w\left[n+\frac{p}{2}\right] \longrightarrow \mathrm{T}^w\left[n+\frac{p}{2}-1\right]$$

has dimension $\frac{1}{2}p(p-1)(p-2)\cdot\ldots\cdot\left(\frac{p}{2}-1\right)$ and we compute the generators.

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Corolary (Gilkey)

Let n be an even integer.
The kernel of the dimensional reduction:

$$\mathrm{T}^{-n-2}\,[n+1] \longrightarrow \mathrm{T}^{-n-2}\,[n]$$

has dimension one, and it is generated by the Lovelock tensor $L_{\frac{n}{2}}$.

Corolary (Gilkey)

Let $n = 2k$ be an even integer.
The kernel of the restriction map

$$\mathrm{T}^{-n}[n] \longrightarrow \mathrm{T}^{-n}[n-1]$$

has dimension one and it is generated by the n-dimensional Pfaffian Pf_n.

Natural tensors can be computed

Theorem

Let $x \in X$ be a point and g_x be a pseudo-Riemannian metric at x. There exists an $\mathbb{R}$-linear isomorphism:

$$\mathrm{T}^w[n] \simeq \bigoplus_{d \in D} \mathrm{Hom}_{O_{g_x}} \left(S^{d_2} N_2 \otimes \cdots \otimes S^{d_r} N_r \, , \, \otimes^p T_x^* X \right)$$

where N is the space of normal tensor D is the set of sequences of nonnegative integers $d = \{d_2, \ldots, d_r\}$ such that:

$$2d_2 + \ldots + r\, d_r = -(p + w) \, . \qquad (1)$$

If such equation has no solutions the such vector space is zero.

The Proof

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So computing natural tensor relies on computing O_n-invariant tensors on a vector space. Just T^*X.

Computing O_n-invariant tensors on a vector space is a classical problem

Theorem (Weyl)

The vector space $\mathrm{Hom}_{O_{p,q}}(E\otimes \overset{m}{\ldots} \otimes E, \mathbb{R})$ of invariant linear forms is zero if m odd; if $m = 2k$ is even, it is generated by total contractions:

$$T_\sigma : e_1 \otimes \ldots \otimes e_{2k} \mapsto g(e_{\sigma(1)}, e_{\sigma(2)}) \cdot \ldots \cdot g(e_{\sigma(2k-1)}, e_{\sigma(2k)})$$

where $\sigma \in S_{2k}$ is a permutation.

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This description of the generators in terms of permutation gives an explicit description of the dimensional reduction.

$$\begin{array}{ccc} \mathrm{T}^{w}[n] & \xrightarrow{r_n} & \mathrm{T}^{w}[n] \\ T_\sigma & \mapsto & T_\sigma \end{array}$$

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