Holonomy and singular foliations

Marco Zambon (Univ. Autónoma Madrid-ICMAT)

joint work with
Iakovos Androulidakis (University of Athens)

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Introduction

We study geometric properties of singular foliations:

A) Is there any sense in which the holonomy groupoid of a singular foliation is smooth?
B) What is the notion of holonomy for a singular foliation?
C) When is a singular foliation isomorphic to its linearization?

For a regular foliation given by an involutive distribution $F \subset TM$, it is well known that:

B) Given a path $\gamma\colon [0,1] \to M$ lying in a leaf, its holonomy is the germ of a diffeomorphism $S_{\gamma(0)} \to S_{\gamma(1)}$ between slices transverse to F.
It is obtained "following nearby paths in leaves of F".

A) The holonomy groupoid is

$$H = \{\text{paths in leaves of } F\}/(\text{holonomy of paths}).$$

It is a Lie groupoid, integrating the Lie algebroid F.

C) Non-invariant Reeb stability theorem:
Suppose L is an embedded leaf and H_x^x is finite
($H_x^x = \{$holonomy of loops based at $x \in L\}$).
Then, nearby L, the foliation F is isomorphic to its linearization.

Singular foliations

Let M be a manifold.
A singular foliation $\mathcal{F}$ is a submodule of the $C^\infty(M)$-module $\mathcal{X}_c(M)$ (the compactly supported vector fields) such that:

- $\mathcal{F}$ is locally finitely generated,
- $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

$(M, \mathcal{F})$ is partitioned into leaves (of varying dimension).

Examples

1) On $M = \mathbb{R}$ take $\mathcal{F}$ to be generated by $x\partial_x$ or by $x^2\partial_x$. Both foliations have the same partition into leaves: $\mathbb{R}_-, \{0\}, \mathbb{R}_+$.
2) On $M = \mathbb{R}^2$ take $\mathcal{F} = \langle \partial_x, y\partial_y \rangle$.
3) If G is a Lie group acting on M, take

$$\mathcal{F} = \langle v_M : v \in \mathfrak{g} \rangle.$$

(Here v_M denotes the infinitesimal generator of the action associated to $v \in \mathfrak{g}$.)
The leaves of $\mathcal{F}$ are the orbits of the action.

A) The holonomy groupoid and smoothness

Let $X_1, \ldots, X_n \in \mathcal{F}$ be local generators of $\mathcal{F}$.
A path holonomy bi-submersion is $(U, \mathbf{s}, \mathbf{t})$ where

$$U \subset M \times \mathbb{R}^n \underset{\mathbf{t}}{\overset{\mathbf{s}}{\rightrightarrows}} M$$

and the (source and target) maps are

$\mathbf{s}(y, \xi) = y$

$\mathbf{t}(y, \xi) = exp_y(\sum_{i=1}^n \xi_i X_i)$, the time-1 flow of $\sum_{i=1}^n \xi_i X_i$ starting at y.

There is a notion of composition and inversion of path holonomy bi-submersions, as well as a notion of morphism.

Take a family of path holonomy bi-submersions $\{U_i\}_{i\in I}$ covering M. Let $\mathcal{U}$ be the family of all finite products of elements of $\{U_i\}_{i\in I}$ and of their inverses.

The holonomy groupoid of the foliation $\mathcal{F}$ [Androulidakis-Skandalis] is

$$H := \coprod_{U\in\mathcal{U}} U/\sim$$

where $u \in U \sim u' \in U'$ if there is a morphism of bi-submersions $f : U \to U'$ (defined near u) such that $f(u) = u'$.

H is a topological groupoid over M, usually not smooth.

Examples 1) Consider the action of S^1 on $M = \mathbb{R}^2$ by rotations. Then

$$H = S^1 \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$$

(the transformation groupoid).
2) Consider the action of $GL(2,\mathbb{R})$ on $M = \mathbb{R}^2$ and the induced foliation. Then

$$H = (\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 - \{0\}) \coprod GL(2,\mathbb{R}).$$

Smoothness of H_L

Let L be a leaf and $x \in L$. There is a short exact sequence of vector spaces

$$0 \to \underbrace{\mathfrak{g}_x}_{\text{a Lie algebra}} \to (\mathcal{F}/I_x\mathcal{F}) \xrightarrow{ev_x} T_xL \to 0$$

where ev_x is evaluation at x.

$$A_L := \cup_{x \in L}(\mathcal{F}/I_x\mathcal{F})$$

is a transitive Lie algebroid over L, with $\Gamma_c(A_L) \cong \mathcal{F}/I_L\mathcal{F}$.

Question: When does A_L integrate to H_L (the restriction of the holonomy groupoid to L)?

Theorem (Debord)

Let $(M, \mathcal{F})$ be a foliation and L a leaf.
The transitive groupoid H_L is smooth and integrates the Lie algebroid A_L.

B) Holonomy

For a regular foliation F and a path γ in a leaf, the holonomy of γ is defined "following nearby paths in the leaves of F".
For singular foliations this fails (think of $M = \mathbb{R}^2$, $\mathcal{F} = \langle x\partial_y - y\partial_x \rangle$, and γ the constant path at the origin).

Question: How to extend the notion of holonomy to singular foliations?

Let $x, y \in (M, \mathcal{F})$ be points in the same leaf L, and fix transversals S_x and S_y.

Theorem

There is a well defined map

$$\Phi_x^y \colon H_x^y \to \frac{GermAut_{\mathcal{F}}(S_x, S_y)}{exp(I_x \mathcal{F}_{S_x})}, \quad h \mapsto \langle \tau \rangle.$$

Here τ is defined as follows, given $h \in H_x^y$:

- *take any bi-submersion $(U, \mathbf{t}, \mathbf{s})$ and $u \in U$ satisfying $[u] = h$,*
- *take any section $\bar{b} \colon S_x \to U$ through u of $\mathbf{s}$ such that $(\mathbf{t} \circ \bar{b})(S_x) \subset S_y$,*

and define $\tau := \mathbf{t} \circ \bar{b} \colon S_x \to S_y$.

Example:
Let $M = \mathbb{R}$ and $\mathcal{F} = \langle x\partial_x \rangle$. We have $H = \mathbb{R} \times M \rightrightarrows M$.
So $H_0^0 \cong \mathbb{R}$, and a transversal S_0 at 0 is a neighborhood of 0 in M. We have:

$$\Phi_0^0(\lambda) = [y \ \mapsto \ e^{\lambda}y] \in \frac{GermAut_{\mathcal{F}}(S_0, S_0)}{exp(I_0 x\partial_x)}.$$

We obtain a groupoid morphism

$$\boxed{\Phi\colon H \to \cup_{x,y} \frac{GermAut_{\mathcal{F}}(S_x, S_y)}{exp(I_x\mathcal{F})_{S_x})}.}$$

Remark: Φ is injective.

Remark: If $\mathcal{F}$ is a regular foliation, then $exp(I_x\mathcal{F}_{S_x}) = \{Id_{S_x}\}$, hence the map Φ recovers the usual notion of holonomy for regular foliations.

The above remarks are two justifications for calling H *holonomy groupoid*.

Linear holonomy

Let L be a leaf.
From the holonomy map Φ we obtain:

1) by taking the derivative of τ:

$$\boxed{\Psi_L \colon H_L \to Iso(NL, NL),}$$

a Lie groupoid representation of H_L on NL.

2) by differentiating Ψ_L:

$$\boxed{\nabla^{L,\perp} \colon A_L \to Der(NL),}$$

the Lie algebroid representation of A_L on NL induced by the Lie bracket.

(Notice that $\Gamma(A_L) = \mathcal{F}/I_L\mathcal{F}$ and $\Gamma(NL) = \mathcal{X}(M)/(\mathcal{F} + I_L\mathcal{X}(M))$.)
Here $\Gamma(Der(NL)) = \{$first order differential operators on $NL\}$.

C) Linearization

Vector field Y on M tangent to $L \rightsquigarrow$
vector field Y_{lin} on NL, defined as follows:

Y_{lin} acts on the fiberwise constant functions as $Y|_L$

Y_{lin} acts on $C^{\infty}_{lin}(NL) \cong I_L/I_L^2$ as $Y_{lin}[f] := [Y(f)]$.

The linearization of $\mathcal{F}$ at L is the foliation $\mathcal{F}_{lin}$ on NL generated by $\{Y_{lin} : Y \in \mathcal{F}\}$.

Lemma

Let L be an embedded leaf.
Then the linearized foliation $\mathcal{F}_{lin}$ is the foliation induced by the Lie groupoid action Ψ_L of H_L on NL.

We say $\mathcal{F}$ is linearizable at L if there is a diffeomorphism mapping $\mathcal{F}$ to $\mathcal{F}_{lin}$.

Remark: When $\mathcal{F} = \langle X \rangle$ with X vanishing at $L = \{x\}$, linearizability of $\mathcal{F}$ means: there is a diffeomorphism taking X to a fX_{lin} for a non-vanishing function f. It is a weaker condition than the linearizability of the vector field X!

Question: When is a singular foliation isomorphic to its linearization?
We don't know, but:

Proposition

Let L be an embedded leaf. Assume that H_x^x is compact for $x \in L$. The following are equivalent:

1) *$\mathcal{F}$ is linearizable about L*
2) *there exists a tubular neighborhood U of L and a (Hausdorff) Lie groupoid $G \rightrightarrows U$, proper at x, inducing the foliation $\mathcal{F}|_U$.*

In that case:
- G can be chosen to be the transformation groupoid of the action Ψ_L of H_L on NL,
- $(U, \mathcal{F}|_U)$ admits the structure of a singular Riemannian foliation.

Bibliography I

- I. Androulidakis and G. Skandalis:
  The holonomy groupoid of a singular foliation.
  J. Reine Angew. Math. **626** (2009), 1–37.

- I. Androulidakis and M. Zambon:
  Smoothness of holonomy covers for singular foliations and essential isotropy.
  `arXiv:1111.1327`, to appear in Math. Z.

- I. Androulidakis and M. Zambon:
  Holonomy transformations for singular foliations.
  `arXiv:1205.6008`

- C. Debord:
  Longitudinal smoothness of the holonomy groupoid.
  Comptes Rendus(2013)

Thank you!