Holonomy and singular foliations

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We study geometric properties of singular foliations:

A) Is there any sense in which the holonomy groupoid of a singular foliation is smooth?
B) What is the notion of holonomy for a singular foliation?
C) When is a singular foliation isomorphic to its linearization?
For a regular foliation given by an involutive distribution $F \subset TM$, it is well known that:

B) Given a path $\gamma : [0, 1] \rightarrow M$ lying in a leaf, its holonomy is the germ of a diffeomorphism $S_{\gamma(0)} \rightarrow S_{\gamma(1)}$ between slices transverse to $F$. It is obtained “following nearby paths in leaves of $F$”. ●

A) The holonomy groupoid is

$$H = \{\text{paths in leaves of } F\}/(\text{holonomy of paths}).$$

It is a Lie groupoid, integrating the Lie algebroid $F$.

C) Non-invariant Reeb stability theorem:
Suppose $L$ is an embedded leaf and $H^x_L$ is finite ($H^x_L = \{\text{holonomy of loops based at } x \in L\}$).
Then, nearby $L$, the foliation $F$ is isomorphic to its linearization.
Singular foliations

Let $M$ be a manifold. A singular foliation $\mathcal{F}$ is a submodule of the $C^\infty(M)$-module $\mathcal{X}_c(M)$ (the compactly supported vector fields) such that:

- $\mathcal{F}$ is locally finitely generated,
- $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

$(M, \mathcal{F})$ is partitioned into leaves (of varying dimension).

Examples

1) On $M = \mathbb{R}$ take $\mathcal{F}$ to be generated by $x \partial_x$ or by $x^2 \partial_x$. Both foliations have the same partition into leaves: $\mathbb{R}_-, \{0\}, \mathbb{R}_+$.

2) On $M = \mathbb{R}^2$ take $\mathcal{F} = \langle \partial_x, y \partial_y \rangle$.

3) If $G$ is a Lie group acting on $M$, take

$$\mathcal{F} = \langle v_M : v \in \mathfrak{g} \rangle.$$ 

(Here $v_M$ denotes the infinitesimal generator of the action associated to $v \in \mathfrak{g}$.)

The leaves of $\mathcal{F}$ are the orbits of the action.
A) The holonomy groupoid and smoothness

Let $X_1, \ldots, X_n \in \mathcal{F}$ be local generators of $\mathcal{F}$. A path holonomy bi-submersion is $(U, s, t)$ where

$$U \subset M \times \mathbb{R}^n \xrightarrow{s} M \xrightarrow{t} M$$

and the (source and target) maps are

$$s(y, \xi) = y$$
$$t(y, \xi) = \exp_y(\sum_{i=1}^n \xi_i X_i), \text{ the time}-1 \text{ flow of } \sum_{i=1}^n \xi_i X_i \text{ starting at } y.$$

There is a notion of composition and inversion of path holonomy bi-submersions, as well as a notion of morphism.
Take a family of path holonomy bi-submersions \( \{U_i\}_{i \in I} \) covering \( M \). Let \( \mathcal{U} \) be the family of all finite products of elements of \( \{U_i\}_{i \in I} \) and of their inverses.

The **holonomy groupoid of the foliation** \( \mathcal{F} \) [Androulidakis-Skandalis] is

\[
H := \coprod_{U \in \mathcal{U}} U / \sim
\]

where \( u \in U \sim u' \in U' \) if there is a morphism of bi-submersions \( f : U \to U' \) (defined near \( u \)) such that \( f(u) = u' \).

\( H \) is a topological groupoid over \( M \), usually not smooth.

**Examples**

1) Consider the action of \( S^1 \) on \( M = \mathbb{R}^2 \) by rotations. Then

\[
H = S^1 \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2
\]

(the transformation groupoid).

2) Consider the action of \( GL(2, \mathbb{R}) \) on \( M = \mathbb{R}^2 \) and the induced foliation. Then

\[
H = (\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 - \{0\}) \coprod GL(2, \mathbb{R}).
\]
Smoothness of $H_L$

Let $L$ be a leaf and $x \in L$. There is a short exact sequence of vector spaces

$$0 \rightarrow \mathfrak{g}_x \rightarrow (\mathcal{F}/I_x\mathcal{F}) \xrightarrow{ev_x} T_xL \rightarrow 0$$

where $ev_x$ is evaluation at $x$.

$$A_L := \bigcup_{x \in L}(\mathcal{F}/I_x\mathcal{F})$$

is a transitive Lie algebroid over $L$, with $\Gamma_c(A_L) \cong \mathcal{F}/I_L\mathcal{F}$.

**Question:** When does $A_L$ integrate to $H_L$ (the restriction of the holonomy groupoid to $L$)?

**Theorem (Debord)**

Let $(M, \mathcal{F})$ be a foliation and $L$ a leaf. The transitive groupoid $H_L$ is smooth and integrates the Lie algebroid $A_L$. 

B) Holonomy

For a regular foliation $F$ and a path $\gamma$ in a leaf, the holonomy of $\gamma$ is defined “following nearby paths in the leaves of $F$”.

For singular foliations this fails (think of $M = \mathbb{R}^2$, $F = \langle x\partial_y - y\partial_x \rangle$, and $\gamma$ the constant path at the origin).

**Question:** How to extend the notion of holonomy to singular foliations?

Let $x, y \in (M, F)$ be points in the same leaf $L$, and fix transversals $S_x$ and $S_y$.

**Theorem**

There is a well defined map

$$
\Phi^y_x : H^y_x \to \frac{\text{GermAut}_F(S_x, S_y)}{\exp(I_x F_{S_x})}, \quad h \mapsto \langle \tau \rangle.
$$

Here $\tau$ is defined as follows, given $h \in H^y_x$:

- take any bi-submersion $(U, t, s)$ and $u \in U$ satisfying $[u] = h$,
- take any section $\bar{b} : S_x \to U$ through $u$ of $s$ such that $(t \circ \bar{b})(S_x) \subset S_y$,

and define $\tau := t \circ \bar{b} : S_x \to S_y$. ✗
Example:
Let $M = \mathbb{R}$ and $\mathcal{F} = \langle x \partial_x \rangle$. We have $H = \mathbb{R} \times M \rightrightarrows M$.
So $H_0^0 \cong \mathbb{R}$, and a transversal $S_0$ at $0$ is a neighborhood of $0$ in $M$. We have:

$$\Phi_0^0(\lambda) = [y \mapsto e^{\lambda}y] \in \frac{\text{GermAut}_\mathcal{F}(S_0, S_0)}{\exp(I_0 x \partial_x)}.$$

We obtain a groupoid morphism

$$\Phi: H \to \bigcup_{x,y} \frac{\text{GermAut}_\mathcal{F}(S_x, S_y)}{\exp(I_x \mathcal{F})_{S_x}}.$$ 

Remark: $\Phi$ is injective.

Remark: If $\mathcal{F}$ is a regular foliation, then $\exp(I_x \mathcal{F}_{S_x}) = \{\text{Id}_{S_x}\}$, hence the map $\Phi$ recovers the usual notion of holonomy for regular foliations.

The above remarks are two justifications for calling $H$ holonomy groupoid.
Linear holonomy

Let $L$ be a leaf. From the holonomy map $\Phi$ we obtain:

1) by taking the derivative of $\tau$:

\[ \Psi_L : H_L \to Iso(NL, NL), \]

a Lie groupoid representation of $H_L$ on $NL$.

2) by differentiating $\Psi_L$:

\[ \nabla^L, \perp : A_L \to Der(NL), \]

the Lie algebroid representation of $A_L$ on $NL$ induced by the Lie bracket.

(Notice that $\Gamma(A_L) = \mathcal{F}/I_L\mathcal{F}$ and $\Gamma(NL) = \mathcal{X}(M)/(\mathcal{F} + I_L\mathcal{X}(M))$.)

Here $\Gamma(Der(NL)) = \{\text{first order differential operators on } NL\}$. 

C) Linearization

Vector field $Y$ on $M$ tangent to $L \leftrightarrow$
vector field $Y_{lin}$ on $NL$, defined as follows:

$Y_{lin}$ acts on the fiberwise constant functions as $Y|_L$

$Y_{lin}$ acts on $C^\infty_{lin}(NL) \cong I_L/I_L^2$ as $Y_{lin}[f] := [Y(f)]$.

The linearization of $\mathcal{F}$ at $L$ is the foliation $\mathcal{F}_{lin}$ on $NL$ generated by
$\{Y_{lin} : Y \in \mathcal{F}\}$.

Lemma

Let $L$ be an embedded leaf.
Then the linearized foliation $\mathcal{F}_{lin}$ is the foliation induced by the Lie groupoid
action $\Psi_L$ of $H_L$ on $NL$. 
We say $\mathcal{F}$ is linearizable at $L$ if there is a diffeomorphism mapping $\mathcal{F}$ to $\mathcal{F}_{lin}$.

Remark: When $\mathcal{F} = \langle X \rangle$ with $X$ vanishing at $L = \{x\}$, linearizability of $\mathcal{F}$ means: there is a diffeomorphism taking $X$ to a $fX_{lin}$ for a non-vanishing function $f$. It is a weaker condition than the linearizability of the vector field $X$!

Question: When is a singular foliation isomorphic to its linearization?
We don’t know, but:

Proposition

Let $L$ be an embedded leaf. Assume that $H^x_x$ is compact for $x \in L$. The following are equivalent:

1) $\mathcal{F}$ is linearizable about $L$

2) there exists a tubular neighborhood $U$ of $L$ and a (Hausdorff) Lie groupoid $G \rightrightarrows U$, proper at $x$, inducing the foliation $\mathcal{F}|_U$.

In that case:
- $G$ can be chosen to be the transformation groupoid of the action $\Psi_L$ of $H_L$ on $NL$,
- $(U, \mathcal{F}|_U)$ admits the structure of a singular Riemannian foliation.
I. Androulidakis and G. Skandalis:  
*The holonomy groupoid of a singular foliation.*  

I. Androulidakis and M. Zambon:  
*Smoothness of holonomy covers for singular foliations and essential isotropy.*  

I. Androulidakis and M. Zambon:  
*Holonomy transformations for singular foliations.*  
arXiv:1205.6008

C. Debord:  
*Longitudinal smoothness of the holonomy groupoid.*  
Comptes Rendus(2013)
Thank you!