Characterization of variational equations on natural bundles

J. Navarro

Department of Mathematics
University of Extremadura, Spain

(joint work with J. B. Sancho)

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Variational principles and Euler-Lagrange equations

Consider functions:

$$\textit{Space} \equiv \left[\begin{array}{c} \text{Smooth sections of a} \\ \text{bundle } F \longrightarrow X \end{array} \right] \longrightarrow \mathbb{R}$$

of the following kind:

$$s \longmapsto \int_X \mathcal{L}\left(x_j, s_i, \frac{\partial s_i}{\partial x_j}, \ldots, \frac{\partial^k s_i}{\partial x^J} \ldots\right) \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

Critical points? $\Rightarrow$ **Euler-Lagrange equations** on s

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$$Space \equiv \left[\begin{array}{c} \text{Smooth functions} \\ \text{on } X \end{array} \right]$$

- Wave equation.
- Heat equation.

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Equation $\rightsquigarrow$ Cohomology class $\rightsquigarrow$ Helmholtz conditions.

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Takens' approach

$\mathbb{T}$ an equation. D a vector field (infinitesimal transformation)

D can generate a conservation law for $\mathbb{T}$.

D is a **symmetry** of $\mathbb{T}$ if

$$L_D \mathbb{T} = 0.$$

Theorem (First Noether's Theorem)

Let $\mathbb{T}$ be a **variational** *equation.*

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Diff(X) is a symmetry of $\mathbb{T}$ $\Leftrightarrow$ the identity $\mathrm{Div}\,\mathbb{T} = 0$ holds

Does the reciprocal holds?

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Takens' statement in the bundle of metrics

$\mathbb{T}$ a 2-tensor (*Ricci*, *Einstein*,...)

Div is the standard divergence operator div.

Takens' statement:

$$\mathbb{T} \text{ es natural y } \mathrm{div}\,\mathbb{T} = 0 \quad \stackrel{?}{\Rightarrow} \quad \mathbb{T} \text{ es variacional}$$

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Theorem (Takens '77)

If $\mathbb{T}$ is of order 2, then Takens' statement holds.

Theorem (Anderson-Pohjanpelto '12)

If $\mathbb{T}$ is of order 3, then Takens' statement holds.

Theorem (N. - Sancho)

If $\dim X = 2$ and $\mathbb{T}$ is of order 4, then Takens' statement holds.

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