Characterization of variational equations on natural bundles

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(joint work with J. B. Sancho)

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1. Variational equations
2. Takens’ problem
3. Takens’ problem in the bundle of metrics
Consider functions:

\[
Space \equiv \left[ \text{Smooth sections of a bundle } F \longrightarrow X \right] \longrightarrow \mathbb{R}
\]

of the following kind:

\[
s \mapsto \int_X \mathcal{L} \left( x_j, s_i, \frac{\partial s_i}{\partial x_j}, \ldots, \frac{\partial^k s_i}{\partial x^j} \ldots \right) \, dx^1 \wedge \ldots \wedge dx^n
\]

Critical points? \Rightarrow \text{Euler-Lagrange equations on } s
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Critical points? \Rightarrow Euler-Lagrange equations on \( \mathbf{s} \)
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Critical points? \Rightarrow Euler-Lagrange equations on \( s \)
Variational principles and Euler-Lagrange equations

Consider functions:

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of the following kind:

\[ s \mapsto -\int_X L \left( x_j, s_i, \frac{\partial s_i}{\partial x_j}, \ldots, \frac{\partial^k s_i}{\partial x^J}, \ldots \right) \, dx^1 \wedge \ldots \wedge dx^n \]

Critical points? \Rightarrow \textbf{Euler-Lagrange equations on } s
Examples

\[ \text{Space} \equiv \left[ \text{Smooth functions on } X \right] \]

- Wave equation.
- Heat equation.

\[ \text{Space} \equiv \left[ \text{Smooth curves on } X \right] \]

- Equation of geodesics.
- Newton’s equations of motion.
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- Einstein field equations.
- Einstein-Maxwell equations.

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Which equations are variational?

- Classical answer:

  \[ \text{Equation} \rightsquigarrow \text{Cohomology class} \rightsquigarrow \text{Helmholtz conditions.} \]
Which equations are variational?

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Takens’ approach

\( T \) an equation. \( D \) a vector field (infinitesimal transformation)

\( D \) can generate a conservation law for \( T \).

\( D \) is a symmetry of \( T \) if

\[ L_D T = 0. \]

**Theorem (First Noether’s Theorem)**

Let \( T \) be a variational equation.

\( D \) is a symmetry of \( T \) \( \iff \) \( D \) generates a conservation law for \( T \).
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Variational equations on natural bundles 
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Space $\equiv \left[ \text{Smooth sections of a natural bundle } F \to X \right]$

Div generalized divergence operator.

$T$ is natural if

$$L_D T = 0 \quad \forall \text{ vector field on } X$$

Theorem (Second Noether’s Theorem for natural bundles)

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Takens’ observation

\[ T \text{ a variational equation} \]

\[ \Downarrow \]

\[ D \text{ is a symmetry of } T \iff D \text{ generates a conservation law of } T \]

or

\[ \text{Diff}(X) \text{ is a symmetry of } T \iff \text{the identity } \text{Div } T = 0 \text{ holds} \]

Does the reciprocal holds?
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Does the reciprocal holds?
Takens’ statement in the bundle of metrics

\[ T \text{ a 2-tensor } (\text{Ricci, Einstein,}...) \]

Div is the standard divergence operator \( \text{div} \).

Takens’ statement:

\[ T \text{ es natural y } \text{div} \ T = 0 \quad \Rightarrow \quad T \text{ es variacional} \]
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Theorem (Takens ‘77)

If $\mathcal{T}$ is of order 2, then Takens’ statement holds.

Theorem (Anderson-Pohjanpelto ‘12)

If $\mathcal{T}$ is of order 3, then Takens’ statement holds.

Theorem (N. - Sancho)

If $\dim X = 2$ and $\mathcal{T}$ is of order 4, then Takens’ statement holds.
**Takens’ statement:**

\[ \mathbb{T} \text{ es natural y } \text{div}\mathbb{T} = 0 \overset{?}{\Rightarrow} \mathbb{T} \text{ es variacional} \]

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