Hipersuperficies espaciales completas con curvatura media constante en un espacio de Robertson-Walker

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2º Congreso de Jóvenes Investigadores de la RSME
Sevilla, 16 ~ 20 Septiembre 2013

Parcialmente financiado por MICINN/FEDER MTM2009-10418
y Fundación Séneca 04540/GERM/06
Calabi-Bernstein theorem (1970)

**Non-parametric version**

The only entire maximal graphs in $\mathbb{L}^3$ are the spacelike planes. Equivalently, the only entire solutions to the maximal surface equation

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Let $M^2$ be a (necessarily complete) Riemannian surface with non-negative Gaussian curvature, $K_M \geq 0$. Then, any complete maximal surface $\Sigma^2$ in $-\mathbb{R} \times M^2$ is totally geodesic. Moreover, if $K_M > 0$ at some point on $M$, then $\Sigma$ is a slice $\{t_0\} \times M$, $\{t_0\} \in \mathbb{R}$. 

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- Consider now a compact Riemannian manifold $M$ (without boundary) and consider any smooth function $f \in C^2(M)$. Then $f$ attains its maximum at some point $p_0 \in M$ and

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Omori-Yau maximum principle (Omori, 1967 and Yau, 1975)

Let \( M^n \) be an \( n \)-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and consider \( f : M^n \rightarrow \mathbb{R} \) a smooth function which is bounded from above on \( M^n \). Then there is a sequence of points \( \{p_k\}_{k \in \mathbb{N}} \subset M^n \) such that

\[
\lim_{k \to \infty} f(p_k) = \sup_M f \quad \text{and} \quad \lim_{k \to \infty} |\nabla f(p_k)| = 0 \quad \text{and} \quad \lim_{k \to \infty} \Delta f(p_k) \leq 0.
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Robertson-Walker spacetimes

Let \((M^n, \langle \cdot, \cdot \rangle_M)\) be a Riemannian manifold of constant sectional curvature \(\kappa\), \(I \subseteq \mathbb{R}\) a real interval and \(f > 0\) a positive smooth function on \(I\):
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- We denote it by $-I \times_f M^n$. 
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\(-I \times_f M^n\) has constant sectional curvature \(\bar{\kappa}\) if and only if \(M^n\) has constant sectional curvature \(\kappa\) and

\[
\frac{f''}{f} = \bar{\kappa} = \frac{(f')^2 + \kappa}{f^2}.
\]
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Some energy conditions

- A spacetime \( \mathcal{M}^{n+1} \) satisfies the **timelike convergence condition (TCC)** if \( \text{Ric}(Z, Z) \geq 0 \), for all timelike vector \( Z \).
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For the case of a Robertson-Walker spacetime:

- $\text{TCC} \iff \kappa \geq \sup I(f f'' - f'f')$
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If $-I \times f |_{\overline{M}^{n+1}}$ has constant sectional curvature it satisfies $\text{NCC}$. 

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$$TCC \iff \begin{cases} \ f'' \leq 0 \\ \kappa \geq \sup_l(\frac{ff''}{l} - f'^2) \end{cases}$$
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- $\text{TCC} \iff f'' \leq 0$

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- $\text{NCC} \iff \kappa \geq \sup_{I} (ff'' - f'^{2})$

$\leadsto$ If $-I \times_f M^n$ has constant sectional curvature it satisfies **NCC**.
Let $\Sigma^n$ be a **spacelike hypersurface** immersed into $-I \times_f M^n$. 

Since $\partial_t = \left(\partial/\partial t\right) (t, x)$ is a unitary timelike vector field globally defined on $-I \times_f M^n$, then there exists a unique timelike unitary normal vector field $N$ globally defined on $\Sigma^n$ such that $\langle N, \partial_t \rangle \leq -1 < 0$ on $\Sigma^n$. $N$ is called the **future-pointing Gauss map** of $\Sigma^n$ and $\cosh \theta = -\langle N, \partial_t \rangle$ measures the normal hyperbolic angle of $\Sigma^n$.

Let $h$ denote the **height function** of $\Sigma^n$, $h = \left(\pi I\right)^\top$. It is not difficult to see that $\nabla h = -\partial_t^\top = -\partial_t + \cosh \theta N$ and $\|\nabla h\|^2 = \cosh^2 \theta - 1$.

Given $t_0 \in \mathbb{R}$, the spacelike hypersurface $\{t_0\} \times M^n$ is called a **slice**. Slices are characterized by $\theta \equiv 0$. Or equivalently, $h = c \in \mathbb{R}$.

Slices are totally umbilical spacelike hypersurfaces with $H = f'(t_0) f(t_0)$.
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Let $A : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ be the **shape operator** of $\Sigma^n$ with respect to $N$ and let $\kappa_1, ..., \kappa_n$ be the principal curvatures of $\Sigma^n$. 

The $r$-mean curvature function $H_r$ of the spacelike hypersurface $\Sigma^n$ is defined by

$$H_r(p) = (-1)^r \sum_{1 \leq i_1 < ... < i_r \leq n} \kappa_{i_1}(p) \cdots \kappa_{i_r}(p), \quad 1 \leq r \leq n.$$ 

In the particular case when $r = 1$, $H_1 = H = -\frac{1}{n} \text{tr}(A)$ is the mean curvature of $\Sigma^n$.

In the case $r = 2$, $H_2$ defines a geometric quantity related to the scalar curvature of the hypersurface. $\Sigma^n$ is said to be contained in a timelike bounded region if it is contained in a slab $\Sigma \subset [t_1, t_2] \times M^n = \{(t, x) \in -I \times_f M^n : t_1 \leq t \leq t_2\}$. 

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Theorem (Parametric version)

Let $-I \times_f M^n$ be a Robertson-Walker spacetime whose Riemannian fiber $M^n$ has constant sectional curvature $\kappa$ and such that it obeys NCC. Let $\Sigma^n$ be a complete spacelike hypersurface of $-I \times_f M^n$ with constant mean curvature $H \neq 0$ and contained in a timelike bounded region. If

$$0 \leq H \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right) \leq H^2$$

and

$$\| \nabla h \|^2 \leq \alpha \left| H - \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right) \right| \beta$$

for some positive constants $\alpha$ and $\beta$, then $\Sigma^n$ is a slice.
Sketch of the proof:

- MAIN IDEA: To apply the O-Y maximum principle to $f(h) \cosh \theta$. 

$\star$ $f(h) \cosh \theta$ is bounded from above since $\Sigma$ is contained in a timelike bounded region and $\|\nabla h\|^2 \leq \alpha |H - \sup_{\Sigma} (f' \circ h)| \beta$.

$\star$ Let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame of $X(\Sigma)$. Then, by the Gauss equation, for any $X \in X(\Sigma)$ it holds that $\text{Ric}(X, X) = \sum_i \langle R(X, E_i) X, E_i \rangle + \|AX + nH^2 X\|^2 \leq \text{Ric}(X, X)$ is bounded $\iff \sum_i \langle R(X, E_i) X, E_i \rangle$ is bounded.
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$$\text{Ric}(X, X) = \sum \langle \overline{R}(X, E_i)X, E_i \rangle + \left\| AX + \frac{nH}{2} X \right\|^2 - \frac{n^2 H^2}{4} \left\| X \right\|^2.$$
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  \]

  $\leadsto$ Ric$(X, X)$ is bounded $\iff$ $\sum_i \langle \bar{R}(X, E_i)X, E_i \rangle$ is bounded.
Sketch of the proof:

We can compute

\[
\sum_i \langle \bar{R}(X, E_i)X, E_i \rangle = \left( \frac{\kappa}{f^2(h)} + \frac{f'^2(h)}{f^2(h)} \right) (n - 1) \|X\|^2 \\
+ \left( \frac{\kappa}{f^2(h)} - (\ln f)''(h) \right) (n - 2) \langle X, \nabla h \rangle^2 \\
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\sum_i \langle \bar{R}(X, E_i) X, E_i \rangle = \left( \frac{\kappa}{f^2(h)} + \frac{f'(h)^2}{f^2(h)} \right) (n - 1) \|X\|^2 \\
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+ \left( \frac{\kappa}{f^2(h)} - (\ln f)''(h) \right) \|X\|^2 \|\nabla h\|^2
\]
Sketch of the proof:

- We can compute

\[
\sum_i \langle \bar{R}(X, E_i)X, E_i \rangle = \left( \frac{\kappa}{f^2(h)} + \frac{f'^2(h)}{f^2(h)} \right) (n - 1)\|X\|^2 \\
+ \left( \frac{\kappa}{f^2(h)} - (\ln f)''(h) \right) (n - 2)\langle X, \nabla h \rangle^2 \\
+ \left( \frac{\kappa}{f^2(h)} - (\ln f)''(h) \right) \|X\|^2\|\nabla h\|^2 \\
\geq \left( \frac{\kappa}{f^2(h)} + \frac{f'^2(h)}{f^2(h)} \right) (n - 1)\|X\|^2
\]
Sketch of the proof:

We can compute

$$\sum_{i} \langle \bar{R}(X, E_i)X, E_i \rangle = \left( \frac{\kappa}{f^2(h)} + \frac{f'^2(h)}{f^2(h)} \right) (n - 1) \|X\|^2$$

$$+ \left( \frac{\kappa}{f^2(h)} - (\ln f)''(h) \right) (n - 2) \langle X, \nabla h \rangle^2$$

$$+ \left( \frac{\kappa}{f^2(h)} - (\ln f)''(h) \right) \|X\|^2 \|\nabla h\|^2$$

$$\geq \left( \frac{\kappa}{f^2(h)} + \frac{f'^2(h)}{f^2(h)} \right) (n - 1) \|X\|^2$$

Since $\kappa$ is constant and $\Sigma^n$ is contained in a timelike bounded region

$$\sum_{i} \langle \bar{R}(X, E_i)X, E_i \rangle$$

is bounded from below.
Sketch of the proof:

There exists a sequence \( \{p_k\}_{k \in \mathbb{N}} \) on \( \Sigma^n \) such that

\[
\lim_{k \to \infty} (f(h(p_k)) \cosh \theta(p_k)) = \sup_{p \in \Sigma^n} (f(h(p)) \cosh \theta(p))
\]

and

\[
\lim_{k \to \infty} \Delta (f(h(p_k)) \cosh \theta(p_k)) \leq 0.
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- On the other hand,

  \[
  \Delta(f(h) \cosh \theta) = - nHf'(h)
  \]

  \[
  + f(h) \cosh \theta \left( n^2 H^2 - n(n - 1)H_2 \right)
  \]

  \[
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On the other hand,

\[
\Delta(f(h) \cosh \theta) \geq - n \cosh \theta f(h) H \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right) \\
+ f(h) \cosh \theta \left( n^2 H^2 - n(n - 1) H_2 \right) \\
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  \]
Sketch of the proof:

There exists a sequence \( \{ p_k \} \subseteq \mathbb{N} \) on \( \Sigma^n \) such that

\[
\lim_{k \to \infty} (f(h(p_k))) \cosh \theta(p_k)) = \sup_{p \in \Sigma^n} (f(h(p))) \cosh \theta(p))
\]

and

\[
\lim_{k \to \infty} \Delta (f(h(p_k))) \cosh \theta(p_k)) \leq 0.
\]

On the other hand,

\[
\Delta (f(h) \cosh \theta) \geq f(h) \cosh \theta nH \left( H - \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right) \right) \\
+ n(n - 1)f(h) \cosh \theta (H^2 - H_2) \\
+ (n - 1)f(h) \cosh \theta \left( \frac{\kappa}{f^2(h)} - (\ln f)'(h) \right) \| \nabla h \|^2
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\geq 0.
\]
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Therefore,

$$0 \geq \lim_{k \to \infty} (\Delta (f(h(p_k)) \cosh \theta(p_k))) \geq 0.$$
Sketch of the proof:

Therefore,

\[ 0 \geq \lim_{k \to \infty} \left( \Delta \left( f(h(p_k)) \cosh \theta(p_k) \right) \right) \geq 0. \]

\[ \implies H \left( H - \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right) \right) = 0. \]
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Therefore,

$$0 \geq \lim_{k \to \infty} (\Delta (f(h(p_k)) \cosh \theta(p_k))) \geq 0.$$

$$\implies H(H - \sup_{\Sigma} (\frac{f'}{f} \circ h)) = 0.$$ 

Since $H \neq 0$, $H - \sup_{\Sigma} (\frac{f'}{f} \circ h) = 0$
Sketch of the proof:

Therefore,

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$$\implies H \left( H - \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right) \right) = 0.$$

Since $H \neq 0$, $H - \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right) = 0$

$$\implies \| \nabla h \|^2 = 0 \implies \Sigma^n \text{ is a slice.}$$
Calabi-Bernstein type results in $-I \times_f M^n$

**Remarks:**

- We do not need to ask $\Sigma^n$ to be contained in a timelike bounded region. We just need that $f$, $f'$ and $f''$ are bounded on $\Sigma^n$ and $\inf_{\Sigma} f(h(p)) > 0$. 

Therefore, for the case of a static Robertson Walker spacetime:

**Corollary**

Let $-I \times_f M^n$ be a static Robertson Walker spacetime whose Riemannian fiber $M^n$ has non-negative constant sectional curvature. Let $\Sigma^n$ be a complete spacelike hypersurface in $-I \times_f M^n$ with constant mean curvature and bounded normal hyperbolic angle. Then, $\Sigma^n$ is maximal.
Remarks:

- We do not need to ask $\Sigma^n$ to be contained in a timelike bounded region. We just need that $f$, $f'$, and $f''$ are bounded on $\Sigma^n$ and $\inf_{\Sigma} f(h(p)) > 0$.

- In order to conclude that $H = \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right)$ we only need to ask the normal hyperbolic angle to be bounded.
**Remarks:**

- We do not need to ask $\Sigma^n$ to be contained in a timelike bounded region. We just need that $f, f'$ and $f''$ are bounded on $\Sigma^n$ and $\inf_{\Sigma} f(h(p)) > 0$.

- In order to conclude that $H = \sup_{\Sigma} \left( \frac{f'}{f} \circ h \right)$ we only need to ask the normal hyperbolic angle to be bounded.

\[ \Rightarrow \] Therefore, for the case of a static Robertson Walker spacetime:

**Corollary**

Let $-I \times M^n$ be a static Robertson Walker spacetime whose Riemannian fiber $M^n$ has non-negative constant sectional curvature. Let $\Sigma^n$ be a complete spacelike hypersurface in $-I \times M^n$ with constant mean curvature and bounded normal hyperbolic angle. Then, $\Sigma^n$ is maximal.
In the particular case when $\Sigma^n$ is an immersed spacelike hypersurface in $\mathbb{L}^{n+1}$, we have $N : \Sigma^n \to \mathbb{H}^n$ where

$$\mathbb{H}^n = \{ x \in \mathbb{L}^{n+1} : \langle x, x \rangle = -1, x_1 \geq 1 \}.$$
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$N(\Sigma)$ is called the **hyperbolic image of $\Sigma^n$**.
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A geodesic ball $B(a, \varrho)$ in $\mathbb{H}^n$ of radius $\varrho > 0$ centered at $a \in \mathbb{H}^n$ is

$$B(a, \varrho) = \{p \in \mathbb{H}^n : -\cosh \varrho \leq \langle p, a \rangle \leq -1\}.$$
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The normal hyperbolic angle of $\Sigma^n$ is bounded if and only if $N(\Sigma) \subseteq B(a, \varrho)$ for certain $a \in \mathbb{H}^n$ and $\varrho > 0$. 

---

Corollary (Aiyama (1992), Xin (1991))

Let $\Sigma^n$ be a complete spacelike hypersurface with constant mean curvature immersed into $\mathbb{L}^{n+1}$. Suppose that the hyperbolic image of $\Sigma^n$ is contained in a geodesic ball of $\mathbb{H}^n$. Then $\Sigma^n$ is a hyperplane.
Calabi-Bernstein type results in \(-I \times_f M^n\)

- In the particular case when \(\Sigma^n\) is an immersed spacelike hypersurface in \(\mathbb{L}^{n+1}\), we have \(N : \Sigma^n \to \mathbb{H}^n\) where

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\]

- \(N(\Sigma)\) is called the hyperbolic image of \(\Sigma^n\).

- A geodesic ball \(B(a, \varrho)\) in \(\mathbb{H}^n\) of radius \(\varrho > 0\) centered at \(a \in \mathbb{H}^n\) is

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\(\Rightarrow\) The normal hyperbolic angle of \(\Sigma^n\) is bounded if and only if \(N(\Sigma) \subseteq B(a, \varrho)\) for certain \(a \in \mathbb{H}^n\) and \(\varrho > 0\).

**Corollary (Aiyama (1992), Xin (1991))**

Let \(\Sigma^n\) be a complete spacelike hypersurface with constant mean curvature immersed into \(\mathbb{L}^{n+1}\). Suppose that the hyperbolic image of \(\Sigma^n\) is contained in a geodesic ball of \(\mathbb{H}^n\). Then \(\Sigma^n\) is a hyperplane.
An **entire vertical graph** in $-I \times_f M^n$ is determined by a smooth function $u \in C^\infty(M)$ and it is given by

$$\Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset -I \times_f M^n.$$
**Entire spacelike vertical graphs in $-I \times_f M^n$**

- An **entire vertical graph** in $-I \times_f M^n$ is determined by a smooth function $u \in C^\infty(M)$ and it is given by
  \[
  \Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset -I \times_f M^n.
  \]
- The metric induced on $M^n$ from the Lorentzian metric of $-I \times_f M^n$ via the graph is
  \[
  \langle , \rangle = -du^2 + f^2(u)\langle , \rangle_{M^n}.
  \]
Entire spacelike vertical graphs in $-I \times_f M^n$

- An entire **vertical graph** in $-I \times_f M^n$ is determined by a smooth function $u \in \mathcal{C}^\infty(M)$ and it is given by
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- The metric induced on $M^n$ from the Lorentzian metric of $-I \times_f M^n$ via the graph is
  $$\langle \cdot , \cdot \rangle = -du^2 + f^2(u)\langle \cdot , \cdot \rangle_{M^n}.$$  

- The graph $\Sigma^n(u)$ is a spacelike hypersurface iff $|Du|^2 < f^2(u)$.  

It is well known that an entire spacelike graph is not necessarily complete. In fact, there exist examples of entire non-complete graphs in $-\mathbb{R} \times H^n$ which are maximal (Albujer, 2008 and Albujer, Alías, 2009) or spacelike with constant mean curvature (Alarcón, Souam).
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- For any entire spacelike graph we have the following relation:
  \[
  \|\nabla h\|^2 = \frac{|Du|^2}{f^2(u) - |Du|^2}.
  \]
Entire spacelike vertical graphs in $-I \times f M^n$

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  \[ \Sigma^n(u) = \{(u(x), x) : x \in M^n\} \subset -I \times f M^n. \]

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An entire vertical graph in \(-I \times f M^n\) is determined by a smooth function \(u \in C^\infty(M)\) and it is given by
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Entire vertical spacelike graphs in $-I \times_f M^n$

- However, it is possible to give a non-parametric version of our main result:

Theorem (Non-parametric version)

Let $-I \times_f M^n$ be a Robertson Walker spacetime whose Riemannian fiber $M^n$ is a complete manifold with constant sectional curvature $\kappa$, and suppose that is obeys NCC. Let $\Sigma^n(u)$ be an entire spacelike graph in $-I \times_f M^n$ with constant mean curvature $H \neq 0$ and contained in a timelike bounded region. If

$$0 \leq H \sup_{x \in \Sigma^n(u)} \frac{f'}{f}(u(x)) \leq H^2$$

and

$$|Du|_{M^n}^2 \leq \frac{\alpha \inf_{\Sigma^n(u)} (f^2(u)) |H - \sup_{\Sigma^n(u)} \frac{f'}{f}(u)|^\beta}{1 + \alpha |H - \sup_{\Sigma^n(u)} \frac{f'}{f}(u)|^\beta}$$

for some positive constants $\alpha$ and $\beta$, then $\Sigma^n(u)$ is a slice.
Sketch of the proof:

- Under the assumptions of the theorem $\Sigma^n(u)$ is a complete hypersurface
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- Under the assumptions of the theorem $\Sigma^n(u)$ is a complete hypersurface

- In fact, for every $X \in \mathfrak{X}(\Sigma)$

\[
\langle X, X \rangle = -\langle Du, X \rangle_{M^n}^2 + f^2(u)\langle X, X \rangle_{M^n}
\]
Sketch of the proof:

- Under the assumptions of the theorem $\Sigma^n(u)$ is a complete hypersurface

- In fact, for every $X \in \mathcal{X}(\Sigma)$

$$\langle X, X \rangle = -\langle Du, X \rangle_{M^n}^2 + f^2(u)\langle X, X \rangle_{M^n} \geq \left( f^2(u) - |Du|^2 \right) \langle X, X \rangle_{M^n}$$
Sketch of the proof:

- Under the assumptions of the theorem $\Sigma^n(u)$ is a complete hypersurface

- In fact, for every $X \in \mathfrak{X}(\Sigma)$

\[
\langle X, X \rangle = -\langle Du, X \rangle^2_{M^n} + f^2(u)\langle X, X \rangle_{M^n} \\
\geq (f^2(u) - |Du|^2) \langle X, X \rangle_{M^n} \\
\geq c\langle X, X \rangle_{M^n}
\]

where $c = \inf_{\Sigma^n(u)} f^2(u)/\left(1 + \alpha|H - \sup_{\Sigma^n(u)} \frac{f'}{f}(u)|^\beta\right) > 0$. 

Therefore $L \geq cL_{M^n}$, where $L$ and $L_{M^n}$ denote the length of a curve on $\Sigma^n(u)$ w.r.t. $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{M^n}$ respectively.

Since $M^n$ is complete by assumption, the induced metric on $\Sigma^n(u)$ from the metric of $-I \times f_{M^n}$ is also complete. The result follows from the parametric version.
Sketch of the proof:

- Under the assumptions of the theorem $\Sigma^n(u)$ is a complete hypersurface

- In fact, for every $X \in \mathfrak{X}(\Sigma)$

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\langle X, X \rangle = -\langle Du, X \rangle_{M^n}^2 + f^2(u)\langle X, X \rangle_{M^n} \\
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$$

where $c = \inf_{\Sigma^n(u)} f^2(u)/\left(1 + \alpha|H - \sup_{\Sigma^n(u)} \frac{f'(u)}{f(u)}|\beta\right) > 0$.

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Sketch of the proof:

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  * In fact, for every $X \in \mathcal{X}(\Sigma)$

    $$\langle X, X \rangle = -\langle Du, X \rangle_{M^n}^2 + f^2(u)\langle X, X \rangle_{M^n} \geq (f^2(u) - |Du|^2) \langle X, X \rangle_{M^n} \geq c\langle X, X \rangle_{M^n}$$

    where $c = \inf_{\Sigma^n(u)} f^2(u)/ (1 + \alpha|H - \sup_{\Sigma^n(u)} f'(u)|^\beta) > 0$.

  \[\implies\] Therefore $L \geq cL_{M^n}$, where $L$ and $L_{M^n}$ denote the length of a curve on $\Sigma^n(u)$ w. r. t. $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{M^n}$ respectively.

  \[\implies\] Since $M^n$ is complete by assumption, the induced metric on $\Sigma^n(u)$ from the metric of $-I \times f M^n$ is also complete.
Sketch of the proof:

**Under the assumptions of the theorem \( \Sigma^n(u) \) is a complete hypersurface**

\[ \langle X, X \rangle = -\langle Du, X \rangle_{M^n}^2 + f^2(u)\langle X, X \rangle_{M^n} \]
\[ \geq (f^2(u) - |Du|^2) \langle X, X \rangle_{M^n} \]
\[ \geq c \langle X, X \rangle_{M^n} \]

where \( c = \inf_{\Sigma^n(u)} f^2(u) / \left( 1 + \alpha |H - \sup_{\Sigma^n(u)} \frac{f'}{f} (u) |^\beta \right) > 0. \)

Therefore \( L \geq cL_{M^n} \), where \( L \) and \( L_{M^n} \) denote the length of a curve on \( \Sigma^n(u) \) w. r. t. \( \langle , \rangle \) and \( \langle , \rangle_{M^n} \) respectively.

Since \( M^n \) is complete by assumption, the induced metric on \( \Sigma^n(u) \) from the metric of \( -I \times f M^n \) is also complete.

The result follows from the parametric version.
¡Gracias por su atención!