Hipersuperficies espaciales completas con curvatura media constante en un espacio de Robertson-Walker

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Calabi-Bernstein theorem (1970)

Non-parametric version

The only entire maximal graphs in $\mathbb{L}^3$ are the spacelike planes.
Equivalently, the only entire solutions to the maximal surface equation

$$\mathrm{Div}\left(\frac{\mathrm{Du}}{\sqrt{1-|\,\mathrm{Du}\,|^2}}\right)=0, \quad |\,\mathrm{Du}\,|^2<1$$

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- ⋆ Cheng and Yau (1976) extended this theorem to general dimension.
- ⋆ Since then several authors have approached this result from different points of view: McNertey (1980), Kobayashi (1983), Estudillo and Romero (1991, '92 y '94), Romero (1996), Alías and Palmer (2001), Albujer and Alías (2009), Romero and Rubio (2010), Caballero, Romero and Rubio (2010).

Two generalizations of the Calabi-Bernstein theorem

Aiyama (1992), Xin (1991)

Let Σ^n be a complete spacelike hypersurface with constant mean curvature immersed into $\mathbb{L}^{n+1}$. Suppose that the hyperbolic image of Σ^n is contained in a geodesic ball of $\mathbb{H}^n$. Then Σ^n is a spacelike hyperplane.

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Let M^2 be a (necessarily complete) Riemannian surface with non-negative Gaussian curvature, $K_M \geq 0$. Then, any complete maximal surface Σ^2 in $-\mathbb{R} \times M^2$ is totally geodesic. Moreover, if $K_M > 0$ at some point on M, then Σ is a slice $\{t_0\} \times M$, $\{t_0\} \in \mathbb{R}$.

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- ⋆ This result is no longer true in $-\mathbb{R} \times \mathbb{H}^2$.

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... Our main tool is the Omori-Yau maximum principle.

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- Consider now a compact Riemannian manifold M (without boundary) and consider any smooth function $f \in \mathcal{C}^2(M)$. Then f attains its maximum at some point $p_0 \in M$ and

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- When M is not compact, a given function $f \in \mathcal{C}^2(M)$ with $\sup_M f < +\infty$ does not necessarily attains its supremum.

The Omori-Yau maximum principle

Omori-Yau maximum principle (Omori, 1967 and Yau, 1975)

Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and consider $f : M^n \to \mathbb{R}$ a smooth function which is bounded from above on M^n. Then there is a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M^n$ such that

$$\lim_{k\to\infty} f(p_k) = \sup_M f \ , \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0 \quad \text{and} \quad \lim_{k\to\infty} \Delta f(p_k) \leq 0.$$

- Let $(M^n, \langle , \rangle_M)$ be a Riemannian manifold of constant sectional curvature κ, $I \subseteq \mathbb{R}$ a real interval and $f > 0$ a positive smooth function on I:

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- A **Robertson-Walker spacetime** is the product manifold $I \times M^n$ endowed with the Lorentzian metric

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⇝ $-I \times_f M^n$ has constant sectional curvature $\overline{\kappa}$ if and only if M^n has constant sectional curvature κ and

$$\frac{f''}{f} = \overline{\kappa} = \frac{(f')^2 + \kappa}{f^2}.$$

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⇝ If $-I \times_f M^n$ has constant sectional curvature it satisfies **NCC**.

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⋆ Slices are totally umbilical spacelike hypersurfaces with $H = \frac{f'(t_0)}{f(t_0)}$.

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- Σ^n is said to be contained in a **timelike bounded region** if it is contained in a slab

$$\Sigma \subset [t_1, t_2] \times M^n = \{(t, x) \in -I \times_f M^n : t_1 \leq t \leq t_2\}.$$

Calabi-Bernstein type results in $-I \times_f M^n$

Theorem (Parametric version)

Let $-I \times_f M^n$ be a Robertson-Walker spacetime whose Riemannian fiber M^n has constant sectional curvature κ and such that it obeys **NCC**. Let Σ^n be a complete spacelike hypersurface of $-I \times_f M^n$ with constant mean curvature $H \neq 0$ and contained in a timelike bounded region. If

$$0 \leq H \sup_\Sigma \left(\frac{f'}{f} \circ h \right) \leq H^2$$

and

$$\|\nabla h\|^2 \leq \alpha \left| H - \sup_\Sigma \left(\frac{f'}{f} \circ h \right) \right|^\beta$$

for some positive constants α and β, then Σ^n is a slice.

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⋆ Let $\{E_1, ...E_n\}$ be a local orthonormal frame of $\mathfrak{X}(\Sigma)$. Then, by the Gauss equation, for any $X \in \mathfrak{X}(\Sigma)$ it holds that

$$\mathrm{Ric}(X,X) = \sum_i \langle \overline{R}(X,E_i)X,E_i)\rangle + \left\| AX + \frac{nH}{2}X \right\|^2 - \frac{n^2H^2}{4}\|X\|^2$$

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$\rightsquigarrow$ $\mathrm{Ric}(X,X)$ is bounded $\Leftrightarrow$ $\sum_i \langle \overline{R}(X,E_i)X,E_i\rangle$ is bounded.

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- We can compute

$$\sum_i \langle \overline{R}(X, E_i)X, E_i \rangle = \left(\frac{\kappa}{f^2(h)} + \frac{f'^2(h)}{f^2(h)} \right) (n-1)\|X\|^2$$
$$+ \left(\frac{\kappa}{f^2(h)} - (\ln f)''(h) \right) (n-2)\langle X, \nabla h \rangle^2$$
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⋆ Since κ is constant and Σ^n is contained in a timelike bounded region

$\rightsquigarrow$ $\sum_i \langle \overline{R}(X,E_i)X,E_i\rangle$ is bounded from below.

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- There exists a sequence $\{p_k\}_{k\in\mathbb{N}}$ on Σ^n such that

$$\lim_{k\to\infty}\left(f(h(p_k))\cosh\theta(p_k)\right)=\sup_{p\in\Sigma^n}\left(f(h(p))\cosh\theta(p)\right)$$

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$$\begin{aligned}\Delta(f(h)\cosh\theta) =& -nHf'(h)\\ &+f(h)\cosh\theta\left(n^2H^2-n(n-1)H_2\right)\\ &+(n-1)f(h)\cosh\theta\left(\frac{\kappa}{f^2(h)}-(\ln f)''(h)\right)\|\nabla h\|^2\end{aligned}$$

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⇝ $\|\nabla h\|^2 = 0$ ⇝ Σ^n is a slice.

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$\rightsquigarrow$ Therefore, for the case of a static Robertson Walker spacetime:

Corollary

Let $-I \times M^n$ be a static Robertson Walker spacetime whose Riemannian fiber M^n has non-negative constant sectional curvature. Let Σ^n be a complete spacelike hypersurface in $-I \times M^n$ with constant mean curvature and bounded normal hyperbolic angle. Then, Σ^n is maximal.

- In the particular case when Σ^n is an immersed spacelike hypersurface in $\mathbb{L}^{n+1}$, we have $N : \Sigma^n \to \mathbb{H}^n$ where

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- A geodesic ball $B(a, \varrho)$ in $\mathbb{H}^n$ of radius $\varrho > 0$ centered at $a \in \mathbb{H}^n$ is

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Corollary (Aiyama (1992), Xin (1991))

Let Σ^n be a complete spacelike hypersurface with constant mean curvature immersed into $\mathbb{L}^{n+1}$. Suppose that the hyperbolic image of Σ^n is contained in a geodesic ball of $\mathbb{H}^n$. Then Σ^n is a hyperplane.

Entire spacelike vertical graphs in $-I \times_f M^n$

- An **entire vertical graph** in $-I \times_f M^n$ is determined by a smooth function $u \in \mathcal{C}^\infty(M)$ and it is given by

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- It is well known that an entire spacelike graph is not necessarily complete.

⋆ In fact, there exist examples of entire non-complete graphs in $-\mathbb{R} \times \mathbb{H}^n$ which are maximal (Albujer, 2008 and Albujer, Alías, 2009) or spacelike with constant mean curvature (Alarcón, Souam).

- However, it is possible to give a non-parametric version of our main result:

Theorem (Non-parametric version)

Let $-I \times_f M^n$ be a Robertson Walker spacetime whose Riemannian fiber M^n is a complete manifold with constant sectional curvature κ, and suppose that is obeys **NCC**. Let $\Sigma^n(u)$ be an entire spacelike graph in $-I \times_f M^n$ with constant mean curvature $H \neq 0$ and contained in a timelike bounded region. If

$$0 \leq H \sup_{x \in \Sigma^n(u)} \frac{f'}{f}(u(x)) \leq H^2$$

and

$$|Du|^2_{M^n} \leq \frac{\alpha \inf_{\Sigma^n(u)}(f^2(u)) \left| H - \sup_{\Sigma^n(u)} \frac{f'}{f}(u) \right|^\beta}{1 + \alpha \left| H - \sup_{\Sigma^n(u)} \frac{f'}{f}(u) \right|^\beta}$$

for some positive constants α and β, then $\Sigma^n(u)$ is a slice.

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- The result follows from the parametric version.

¡Gracias por su atención!