Conjugación Galois de superficies isógenas a un producto

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(Joint work with Gabino González-Diez)

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There are topological invariants (for instance the Betti numbers $\dim H_i(V)$). However, there are varieties with the same Betti numbers which are NOT homeomorphic.

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be an algebraic curve. We define:

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$$\Rightarrow \; genus(C) = h^0(\Omega_C) = h^0(\Omega_{C^\sigma}) = genus(C^\sigma) \; \Rightarrow$$
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where $\{F_\alpha\} \subset \mathbb{C}[x_1, \ldots, x_n]$ is a finite set.

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Several other examples have been constructed since then.

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(F. Catanese, '00)

One can construct Beauville surfaces in a purely group theoretical way.
A Beauville surface structure on G is determined by two triples of generators (a_1, b_1, c_1) and (a_2, b_2, c_2) of G or orders (p_1, q_1, r_1) and (p_2, q_2, r_2)

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(i) $a_1 b_1 c_1 = a_2 b_2 c_2 = 1$;
(ii) $1/p_i + 1/q_i + 1/r_i < 1$;
(iii) $\Sigma(a_1, b_1, c_1) \cap \Sigma(a_2, b_2, c_2) = \{1_G\}$, where we define

$$\Sigma(a, b, c) = \bigcup_{g \in G} \bigcup_{j \in \mathbb{N}} \{g a^j g^{-1}, \; g b^j g^{-1}, \; g c^j g^{-1}\}$$

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Each triple (a_i, b_i, c_i) defines one of the curves C_i and an action of G such that:

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We say that the corresponding Beauville surface has type $(p_1, q_1, r_1; p_2, q_2, r_2)$.

Beauville surfaces are good candidates to find our examples of non-homeomorphic Galois conjugate varieties.

Rigidity Theorem (F. Catanese, '00;)

Let $S = (C_1 \times C_2)/G$ and $S' = (C'_1 \times C'_2)/G'$ be Beauville surfaces. If S and S' are homeomorphic, then

$$\begin{cases} C'_1 \cong C_1 \text{ or } \overline{C_1} \\ \quad\quad \text{and} \\ C'_2 \cong C_2 \text{ or } \overline{C_2} \end{cases}$$

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Now, Beauville surfaces are complex algebraic surfaces (even defined over $\overline{\mathbb{Q}}$!). For $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ one has

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<u>PROBLEM</u>: We don't usually have algebraic equations for the curves C_1 and C_2. The few known ones are defined over $\mathbb{Q}$ (C_i = Fermat curve) $\Rightarrow$ $S^\sigma = S$.
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...fortunately in some case this can be done (trust me!).

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- S defined by the triples

$$a_1 = \begin{pmatrix} 1 & 5 \\ 4 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}, \quad c_1 = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix}$$

$$a_2 = \begin{pmatrix} 6 & 1 \\ 3 & 3 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 3 & 3 \\ 4 & 2 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- S' defined by the triples

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Using the previous result (the one you trusted was true!) one can prove that if $\sigma(\xi_8) = \xi_8^5$ then $C_1^\sigma \not\cong C_1$ and $C_1^\sigma \not\cong \overline{C_1}$.

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Therefore, by Catanese's rigidity results one has:

Orbits of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (G. González-Diez, –,'12)

If $\sigma(\xi_8) = \xi_8^5$ then S^σ is not homeomorphic to S. The orbit of S consists of the two Beauville surfaces S and S' defined above.

These tools can be applied to other groups to get similar results.

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Theorem (G. González-Diez, –,'12)

Let $p \geq 13$ be a prime number and let $n > 6$ be a divisor either of $(p-1)/2$ or of $(p+1)/2$. There exists a Beauville surface S with Beauville group $G = \mathrm{PSL}(2,p)$ whose orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains at least $\phi(n)/2$ surfaces with mutually non-isomorphic fundamental groups.

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Let $p \geq 13$ *be a prime number and let* $n > 6$ *be a divisor either of* $(p-1)/2$ *or of* $(p+1)/2$*. There exists a Beauville surface* S *with Beauville group* $G = \mathrm{PSL}(2,p)$ *whose orbit under the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *contains at least* $\phi(n)/2$ *surfaces with mutually non-isomorphic fundamental groups.*

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Theorem (G. Jones)

Let $p \equiv 19 \bmod (24)$ be a prime number and set $m = (p^2-1)/2$. There exists a Beauville surface S with Beauville group $G = \mathrm{PGL}(2,p)$ whose orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains exactly $\phi(m)/4$ surfaces with mutually non-isomorphic fundamental groups.

These tools can be applied to other groups to get similar results.

Theorem (G. González-Diez, –,'12)

Let $p \geq 13$ be a prime number and let $n > 6$ be a divisor either of $(p-1)/2$ or of $(p+1)/2$. There exists a Beauville surface S with Beauville group $G = \mathrm{PSL}(2, p)$ whose orbit under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ contains at least $\phi(n)/2$ surfaces with mutually non-isomorphic fundamental groups.

Theorem (G. González-Diez, G. Jones, –)

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¡Gracias!