The forward problem for the electromagnetic Helmholtz equation with critical singularities

Miren Zubeldia

(joint with J. A. Barceló and L. Vega)

University of Helsinki

2cji 2013, Sevilla

19th September 2013

Contents

Contents

Contents

Contents

The Helmholtz equation: main questions

$$\Delta u(x) + k^2 u(x) = f(x), \qquad x \in \mathbb{R}^d.$$

The Helmholtz equation: main questions

$$\Delta u(x) + k^2 u(x) = f(x), \qquad x \in \mathbb{R}^d.$$

The forward problem for the Helmholtz equation:

The Helmholtz equation: main questions

$$\Delta u(x) + k^2 u(x) = f(x), \qquad x \in \mathbb{R}^d.$$

The forward problem for the Helmholtz equation:

1. Existence and uniqueness of solution?

The Helmholtz equation: main questions

$$\Delta u(x) + k^2 u(x) = f(x), \qquad x \in \mathbb{R}^d.$$

The forward problem for the Helmholtz equation:

1. Existence and uniqueness of solution?

2. Existence of the far field pattern/scattering cross-section?

The Helmholtz equation: main questions

$$\Delta u(x) + k^2 u(x) = f(x), \qquad x \in \mathbb{R}^d.$$

The forward problem for the Helmholtz equation:

1. Existence and uniqueness of solution?
2. Existence of the far field pattern/scattering cross-section?
3. Spectral representation of Δ?

The Helmholtz equation: main questions

$$\Delta u(x) + k^2 u(x) = f(x), \qquad x \in \mathbb{R}^d.$$

The forward problem for the Helmholtz equation:

1. Existence and uniqueness of solution?

2. Existence of the far field pattern/scattering cross-section?

3. Spectral representation of Δ?

Key: Resolvent estimates

$$R(k^2) = (\Delta + k^2)^{-1}$$

Existence and uniqueness of solution

Existence: Limiting absorption principle

$$(\Delta + k^2 \pm i\varepsilon)u_{\pm\varepsilon} = f, \qquad \varepsilon \neq 0$$

Existence and uniqueness of solution

Existence: Limiting absorption principle

$$(\Delta + k^2 \pm i\varepsilon)u_{\pm\varepsilon} = f, \qquad \varepsilon \neq 0$$

$$u_{\pm} = \lim_{\varepsilon \to \pm 0} u_{\pm\varepsilon} = (\Delta + k^2 \pm i0)^{-1} f$$

Existence and uniqueness of solution

Existence: Limiting absorption principle

$$(\Delta + k^2 \pm i\varepsilon)u_{\pm\varepsilon} = f, \qquad \varepsilon \neq 0$$

$$u_{\pm} = \lim_{\varepsilon \to \pm 0} u_{\pm\varepsilon} = (\Delta + k^2 \pm i0)^{-1} f$$

Uniqueness result:

$$\Delta u_{\pm} + k^2 u_{\pm} = 0 \quad + \quad \underbrace{(\text{S.R.C})}_{\downarrow} \quad \Longrightarrow \quad u_{\pm} = 0$$

Sommerfeld radiation conditions:

$$\lim_{|x| \to +\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial u_{\pm}}{\partial |x|} \mp iku_{\pm} \right) = 0.$$

Existence and uniqueness of solution

Existence: Limiting absorption principle

$$(\Delta + k^2 \pm i\varepsilon)u_{\pm\varepsilon} = f, \qquad \varepsilon \neq 0$$

$$u_\pm = \lim_{\varepsilon \to \pm 0} u_{\pm\varepsilon} = (\Delta + k^2 \pm i0)^{-1} f$$

Uniqueness result:

$$\Delta u_\pm + k^2 u_\pm = 0 \quad + \quad \underbrace{(\text{S.R.C})}_{\downarrow} \quad \Longrightarrow \quad u_\pm = 0$$

Sommerfeld radiation conditions:

$$\lim_{|x| \to +\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial u_\pm}{\partial |x|} \mp iku_\pm \right) = 0.$$

Outgoing solution: $u_+ := u = R(k^2)f, \quad R(k^2) := (\Delta + k^2 + i0)^{-1}$ resolvent operator.

Far field pattern and cross-section

Agmon-Hörmander ('76): Limiting absorption principle and asymptotic properties of the outgoing solution.

Far field pattern and cross-section

Agmon-Hörmander ('76): Limiting absorption principle and asymptotic properties of the outgoing solution.

$$u(x) = c_d k^{\frac{d-1}{2}} \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} u_\infty\left(k, \frac{x}{|x|}\right) + o(|x|^{-\frac{(d-1)}{2}}), \qquad |x| \to \infty.$$

- u_∞ is the far field pattern or scattering amplitude of u:

Far field pattern and cross-section

Agmon-Hörmander ('76): Limiting absorption principle and asymptotic properties of the outgoing solution.

$$u(x) = c_d k^{\frac{d-1}{2}} \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} u_\infty \left(k, \frac{x}{|x|}\right) + o(|x|^{-\frac{(d-1)}{2}}), \qquad |x| \to \infty.$$

- u_∞ is the far field pattern or scattering amplitude of u:

$$u_\infty(k,\omega) := g_{k^2}(\omega) := \lim_{|x|\to\infty} c_{d,k} |x|^{\frac{d-1}{2}} e^{-ik|x|} u(\omega|x|),$$

in $L^2(S^{d-1})$, $\omega = \frac{x}{|x|}$.

Far field pattern and cross-section

Agmon-Hörmander ('76): Limiting absorption principle and asymptotic properties of the outgoing solution.

$$u(x) = c_d k^{\frac{d-1}{2}} \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} u_\infty\left(k, \frac{x}{|x|}\right) + o(|x|^{-\frac{(d-1)}{2}}), \qquad |x| \to \infty.$$

- u_∞ is the far field pattern or scattering amplitude of u:

$$u_\infty(k,\omega) := g_{k^2}(\omega) := \lim_{|x|\to\infty} c_{d,k}|x|^{\frac{d-1}{2}} e^{-ik|x|} u(\omega|x|),$$

in $L^2(S^{d-1})$, $\omega = \frac{x}{|x|}$.

- Scattering cross-sections: absolute values of the far field pattern.

$$\mathcal{G}_{k^2}(\omega) = c_{d,k}^2 \lim_{|x|\to\infty} |x|^{d-1}|u(\omega|x|)|^2 \qquad \omega \in S^{d-1} \quad \text{in} \quad L^1(S^{d-1}).$$

Resolvent estimates

Agmon-Hörmander ('76): $|||R(k^2)f|||_1 \leq C|k|^{-1}N_1(f)$.

$$|||u|||_1 := \sup_{R>1} \left(\frac{1}{R} \int_{|x|\leq R} |u(x)|^2 dx \right)^{\frac{1}{2}}$$

$$N_1(f) := \sum_{j>0} \left(2^{j+1} \int_{C(j)} |f(x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{|x|\leq 1} |f(x)|^2 dx \right)^{\frac{1}{2}},$$

where $C(j) = \{x \in \mathbb{R}^d : 2^{j-1} \leq |x| \leq 2^j\}$.

Resolvent estimates

Agmon-Hörmander ('76): $|||R(k^2)f|||_1 \leq C|k|^{-1}N_1(f)$.

$$|||u|||_1 := \sup_{R>1}\left(\frac{1}{R}\int_{|x|\leq R}|u(x)|^2dx\right)^{\frac{1}{2}}$$

$$N_1(f) := \sum_{j>0}\left(2^{j+1}\int_{C(j)}|f(x)|^2dx\right)^{\frac{1}{2}} + \left(\int_{|x|\leq 1}|f(x)|^2dx\right)^{\frac{1}{2}},$$

where $C(j) = \{x \in \mathbb{R}^d : 2^{j-1} \leq |x| \leq 2^j\}$.

$$\int uf \leq |||u|||_1 N_1(f)$$

Contents

The magnetic Schrödinger operator

$$H_A = (\nabla + iA)^2 + V \quad \longrightarrow \quad H_A = \nabla_A^2 + V, \qquad \nabla_A := \nabla + iA.$$

$A : \mathbb{R}^d \to \mathbb{R}^d$ the magnetic vector potential.

$V : \mathbb{R}^d \to \mathbb{R}$ the electric scalar potential.

The magnetic Schrödinger operator

$$H_A = (\nabla + iA)^2 + V \quad \longrightarrow \quad H_A = \nabla_A^2 + V, \quad \nabla_A := \nabla + iA.$$

$A : \mathbb{R}^d \to \mathbb{R}^d$ the magnetic vector potential.

$V : \mathbb{R}^d \to \mathbb{R}$ the electric scalar potential.

- The magnetic field: a $d \times d$ anti-symmetric matrix defined by

$$B = (DA) - (DA)^t, \quad B_{kj} = \left(\frac{\partial A_k}{\partial x_j} - \frac{\partial A_j}{\partial x_k} \right) \quad k, j = 1, \ldots, d.$$

- In dimension $d = 3$, B is identified by the vector field *curl* A via the vector product

$$Bv = curl\, A \times v, \quad \forall v \in \mathbb{R}^3.$$

The electromagnetic Helmholtz equation

We are interested in the study of the equation

$$(H_A + \lambda)u \;=\; (\nabla + iA)^2 u + Vu + \lambda u = f, \quad \lambda \in \mathbb{R}.$$

The electromagnetic Helmholtz equation

We are interested in the study of the equation

$$(H_A+\lambda)u \;=\; (\nabla+iA)^2u+Vu+\lambda u=f, \quad \lambda\in\mathbb{R}.$$

Self-adjointness of H_A in $L^2(\mathbb{R}^d)$

$$A_j\in L^2_{loc}, \quad \int|V||u|^2\le\nu\int|\nabla_A u|^2+C_\nu\int|u|^2 \qquad 0<\nu<1$$

The electromagnetic Helmholtz equation

We are interested in the study of the equation

$$(H_A + \lambda)u \;=\; (\nabla + iA)^2 u + Vu + \lambda u = f, \quad \lambda \in \mathbb{R}.$$

Self-adjointness of H_A in $L^2(\mathbb{R}^d)$

$$A_j \in L^2_{loc}, \quad \int |V||u|^2 \le \nu \int |\nabla_A u|^2 + C_\nu \int |u|^2 \qquad 0 < \nu < 1$$

$$D(H_A) =: H^1_A(\mathbb{R}^d) = \{\phi \in L^2(\mathbb{R}^d) : \int |\nabla_A \phi|^2 < \infty\}.$$

Literature

- When $A = 0$: the electric case. Well studied topic (Agmon, Ikebe, Isozaki, Kato, Mourre, Perthame-Vega, ...)

Literature

- When $A = 0$: the electric case. Well studied topic (Agmon, Ikebe, Isozaki, Kato, Mourre, Perthame-Vega, . . .)

- If $A \neq 0$, results for regular potentials, decay at infinity, local singularities.

Literature

- When $A = 0$: the electric case. Well studied topic (Agmon, Ikebe, Isozaki, Kato, Mourre, Perthame-Vega, ...)

- If $A \neq 0$, results for regular potentials, decay at infinity, local singularities.

  - The forward problem for H_A: **Agmon-Hörmander ('76)** (Perturbative techniques)
  - Limiting absorption principle: **Ikebe-Saito ('72), Saito ('87), ...** (Multiplier techniques)
  - Far field pattern ($A_j \in C^2(\mathbb{R}^d)$): **Iwatsuka ('82)**
  - Resolvent estimates: **Fanelli ('09)**, **Mochizuki ('11)**, **Barceló-Fanelli-Ruiz-Vilela ('11) ...**

Multiplier techniques based on integration by parts

Main tool: **Morawetz ('68), Ikebe-Saito ('72) Perthame-Vega ('99), Fanelli ('09) . . .**

Multiplier techniques based on integration by parts

Main tool: **Morawetz ('68), Ikebe-Saito ('72) Perthame-Vega ('99), Fanelli ('09) . . .**

$$(\nabla + iA)^2 u + Vu + \lambda u + i\varepsilon u = f, \quad \lambda \in \mathbb{R},\ \varepsilon > 0.$$

Multiplier techniques based on integration by parts

Main tool: **Morawetz ('68), Ikebe-Saito ('72) Perthame-Vega ('99), Fanelli ('09) ...**

$$(\nabla + iA)^2 u + Vu + \lambda u + i\varepsilon u = f, \quad \lambda \in \mathbb{R}, \ \varepsilon > 0.$$

Symmetric multiplier: $\varphi \bar{u}, \quad \varphi : \mathbb{R}^d \to \mathbb{R}.$

Anti-symmetric multiplier: $\nabla \psi \cdot \overline{\nabla_A u} + \frac{1}{2} \Delta \psi \bar{u}, \quad \psi : \mathbb{R}^d \to \mathbb{R}.$

Multiplier techniques based on integration by parts

Main tool: **Morawetz ('68), Ikebe-Saito ('72) Perthame-Vega ('99), Fanelli ('09) . . .**

$$(\nabla + iA)^2 u + Vu + \lambda u + i\varepsilon u = f, \quad \lambda \in \mathbb{R},\ \varepsilon > 0.$$

Symmetric multiplier: $\varphi \bar{u}, \quad \varphi : \mathbb{R}^d \to \mathbb{R}.$

Anti-symmetric multiplier: $\nabla\psi \cdot \overline{\nabla_A u} + \frac{1}{2}\Delta\psi \bar{u}, \quad \psi : \mathbb{R}^d \to \mathbb{R}.$

We use the Leibnitz formula $\nabla_A(fg) = (\nabla_A f)g + f(\nabla g)$.

Multiplier techniques based on integration by parts

Main tool: **Morawetz ('68), Ikebe-Saito ('72) Perthame-Vega ('99), Fanelli ('09) . . .**

$$(\nabla + iA)^2 u + Vu + \lambda u + i\varepsilon u = f, \quad \lambda \in \mathbb{R}, \varepsilon > 0.$$

Symmetric multiplier: $\varphi \bar{u}, \quad \varphi : \mathbb{R}^d \to \mathbb{R}.$

Anti-symmetric multiplier: $\nabla \psi \cdot \overline{\nabla_A u} + \frac{1}{2} \Delta \psi \bar{u}, \quad \psi : \mathbb{R}^d \to \mathbb{R}.$

We use the Leibnitz formula $\nabla_A(fg) = (\nabla_A f)g + f(\nabla g)$.

Regular solution:

$$u \in H^1_A(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \nabla_A u \in L^2(\mathbb{R}^d)\}.$$

Multiplier techniques based on integration by parts

Main tool: **Morawetz ('68), Ikebe-Saito ('72) Perthame-Vega ('99), Fanelli ('09) . . .**

$$(\nabla + iA)^2 u + Vu + \lambda u + i\varepsilon u = f, \quad \lambda \in \mathbb{R}, \ \varepsilon > 0.$$

Symmetric multiplier: $\varphi \bar{u}, \quad \varphi : \mathbb{R}^d \to \mathbb{R}.$

Anti-symmetric multiplier: $\nabla\psi \cdot \overline{\nabla_A u} + \frac{1}{2}\Delta\psi \bar{u}, \quad \psi : \mathbb{R}^d \to \mathbb{R}.$

We use the Leibnitz formula $\nabla_A(fg) = (\nabla_A f)g + f(\nabla g)$.

Regular solution:

$$u \in H^1_A(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \nabla_A u \in L^2(\mathbb{R}^d)\}.$$

Singular magnetic and electric potentials.

Contents

The forward problem

$$(\nabla + iA)^2 u + Vu + \lambda u = f, \qquad (\lambda \in \mathbb{R})$$

with critical singularities on the potentials:

$$V = \frac{\nu_1}{|x|^2}, \qquad |B| \leq \frac{\nu_2}{|x|^2}, \qquad \nu_1, \nu_2 > 0 \quad \text{small} \quad \text{(sharp)}.$$

The forward problem

$$(\nabla + iA)^2 u + Vu + \lambda u = f, \qquad (\lambda \in \mathbb{R})$$

with critical singularities on the potentials:

$$V = \frac{\nu_1}{|x|^2}, \qquad |B| \le \frac{\nu_2}{|x|^2}, \qquad \nu_1, \nu_2 > 0 \quad \text{small} \quad \text{(sharp)}.$$

- $\lambda A(\lambda x) = A(x)$ and $x \cdot A(x) = 0$.

The forward problem

$$(\nabla + iA)^2 u + Vu + \lambda u = f, \qquad (\lambda \in \mathbb{R})$$

with critical singularities on the potentials:

$$V = \frac{\nu_1}{|x|^2}, \qquad |B| \leq \frac{\nu_2}{|x|^2}, \qquad \nu_1, \nu_2 > 0 \quad \text{small} \quad \text{(sharp)}.$$

- $\lambda A(\lambda x) = A(x)$ and $x \cdot A(x) = 0$.

- **Main ingredients:** Uniform resolvent estimates, Sommerfeld radiation condition for solutions $u \in H^1_A(\mathbb{R}^d)$ of the equation

$$\nabla_A^2 u + Vu + \lambda u + i\varepsilon u = f.$$

Key resolvent estimates

Theorem

For $d \geq 2$ and for any $\lambda \in \mathbb{R}$,

$$\int |\nabla_A (e^{-i\lambda^{1/2}|x|} u)|^2 \leq C \int |x|^2 |f|^2, \qquad 0 < \varepsilon < \lambda$$

$$\int |\nabla_A u|^2 \leq C \int |x|^2 |f|^2, \qquad \lambda \leq \varepsilon.$$

Key resolvent estimates

Theorem

For $d \geq 2$ and for any $\lambda \in \mathbb{R}$,

$$\int |\nabla_A(e^{-i\lambda^{1/2}|x|}u)|^2 \leq C \int |x|^2|f|^2, \qquad 0 < \varepsilon < \lambda$$

$$\int |\nabla_A u|^2 \leq C \int |x|^2|f|^2, \qquad \lambda \leq \varepsilon.$$

As a consequence, by the magnetic Hardy inequality and the diamagnetic inequality ,

$$\int \frac{|u|^2}{|x|^2} \leq C \int |x|^2|f|^2.$$

Applications

- Limiting absorption principle with critical singularities for $d \geq 2$ and all $\lambda \in \mathbb{R}$.

Applications

- Limiting absorption principle with critical singularities for $d \geq 2$ and all $\lambda \in \mathbb{R}$.
- Existence of the cross section? $\rightarrow$ there exists a function $\mathcal{G}_\lambda \in L^1(S^{d-1})$ such that

$$\lim_{r\to\infty} \int_{|x|=1} \left| r^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r} u(r\omega) d\sigma(\omega) \right|^2 = \int_{|x|=1} \mathcal{G}_\lambda(\omega) d\sigma(\omega)?$$

$\omega = \frac{x}{|x|}$, $r = |x|$

Key SRC + AH estimate

Theorem

For $d \geq 3$ and for any $\lambda \geq \lambda_0 > 0$,

$$\sup_{R\geq 1} R \int_{|x|\geq R} |\nabla_A(e^{-i\sqrt{\lambda}|x|}u)|^2 \leq C \left[\int |x|^3|f|^2 + (N_1(f))^2 \right],$$

where $C = C(\lambda_0) > 0$.

Key SRC + AH estimate

Theorem

For $d \geq 3$ and for any $\lambda \geq \lambda_0 > 0$,

$$\sup_{R \geq 1} R \int_{|x| \geq R} |\nabla_A (e^{-i\sqrt{\lambda}|x|} u)|^2 \leq C \left[\int |x|^3 |f|^2 + (N_1(f))^2 \right],$$

where $C = C(\lambda_0) > 0$.

$$\lambda |||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C (N_1(f))^2$$

Key SRC + AH estimate

Theorem

For $d \geq 3$ and for any $\lambda \geq \lambda_0 > 0$,

$$\sup_{R\geq 1} R \int_{|x|\geq R} |\nabla_A (e^{-i\sqrt{\lambda}|x|}u)|^2 \leq C \left[\int |x|^3 |f|^2 + (N_1(f))^2 \right],$$

where $C = C(\lambda_0) > 0$.

$$\lambda |||u|||_1^2 + |||\nabla_A u|||_1^2 \leq C(N_1(f))^2$$

$$|B| + |V| \leq \begin{cases} \frac{C}{|x|^{2-\alpha}} & \text{if } |x| \leq 1,\ d = 3 \\ \frac{c}{|x|^2} & \text{if } |x| \leq 1,\ d > 3 \\ \frac{C}{|x|^{3+\alpha}} & \text{if } |x| \geq 1 \end{cases}$$

Conclusions

- Existence and uniqueness of the cross section:

$$\mathcal{G}_\lambda(\omega) = \lim_{r\to\infty} \left| r^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r} u(r\omega) \right|^2 \quad \text{in} \quad L^1(S^{d-1}), \quad \omega = \frac{x}{|x|}.$$

$$\left(\mathcal{G}_\lambda(\omega) = |g_\lambda(\omega)|^2, \qquad g_\lambda \text{ far field pattern} \right)$$

Conclusions

- Existence and uniqueness of the cross section:

$$\mathcal{G}_\lambda(\omega) = \lim_{r\to\infty} \left| r^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r} u(r\omega) \right|^2 \quad \text{in} \quad L^1(S^{d-1}), \quad \omega = \frac{x}{|x|}.$$

$$\left(\mathcal{G}_\lambda(\omega) = |g_\lambda(\omega)|^2, \qquad g_\lambda \text{ far field pattern} \right)$$

- Spectral properties of $H_A = \nabla_A^2 + V$:

Conclusions

- Existence and uniqueness of the cross section:

$$\mathcal{G}_\lambda(\omega) = \lim_{r\to\infty} \left| r^{\frac{d-1}{2}} e^{-i\lambda^{1/2} r} u(r\omega) \right|^2 \quad \text{in} \quad L^1(S^{d-1}), \quad \omega = \frac{x}{|x|}.$$

$$\left(\mathcal{G}_\lambda(\omega) = |g_\lambda(\omega)|^2, \qquad g_\lambda \text{ far field pattern} \right)$$

- Spectral properties of $H_A = \nabla_A^2 + V$:

Let $E(0,\infty)$ be the projection onto the positive part of the spectrum of $-H_A$. Then

$$(E(0,\infty)f, f) = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} \|\mathcal{G}_\lambda(\omega)\|_{L^1(S^{d-1})}\, d\lambda.$$

The end

Thank you!

Eskerrikasko!!