Existencia de solución para un problema de Dirichlet casi lineal con crecimiento cuadrático en el gradiente

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Problema bajo estudio

$$\begin{cases} -\mathrm{div}[\,(a(x) + |u|^q)\,\nabla u\,] + b(x)\,u\,|u|^{p-1}\,|\nabla u|^2 = f(x)\,, & x \in \Omega\,. \\ u(x) = 0\,, & x \in \partial\Omega\,, \end{cases} \qquad (P)$$

- $\Omega \subset \mathbb{R}^N$ abierto y acotado;
- $0 < \alpha \leq a(x) \leq \beta$, $0 < \mu \leq b(x) \leq \nu$;
- $p\,, q > 0$, $f \in L^1(\Omega)$.

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Buscamos soluciones de energía finita

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El lower order term verifica la condición de signo

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Condición de signo

$$u\;\;b\,u\,|u|^{p-1}\,|\nabla u|^2\ge 0$$

Problema bajo estudio

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Condición de signo

$$u\;\; b\,u\,|u|^{p-1}\,|\nabla u|^2 \ge 0$$

[Boccardo, Gallouët, 1992] («efecto regularizante del lower order term»)

Resultados previos

$$\begin{cases} -\mathrm{div}[\,(a(x)+|u|^q)\ \nabla u\,] + b(x)\,u\,|u|^{p-1}\,|\nabla u|^2 = f(x)\,, & x \in \Omega\,, \\ u(x) = 0\,, & x \in \partial\Omega\,. \end{cases} \qquad (P)$$

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[Boccardo, 2011]

Si $\mathbf{f \geq 0}$, $\mathbf{p \geq 2\,q} \Longrightarrow \exists\, u \in H_0^1(\Omega)$ sol. positiva de (P) :

$$b(x)\,u^p\,|\nabla u|^2 \in L^1(\Omega)\,,$$

$$\int_\Omega (a(x)+u^q)\,\nabla u \nabla\varphi + \int_\Omega b(x)\,u^p|\nabla u|^2\,\varphi = \int_\Omega f\,\varphi\,,\ \ \forall\varphi\in H_0^1(\Omega)\cap L^\infty(\Omega)\,.$$

Resultado principal

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Teorema

Si $p\,,q>0 \Longrightarrow \exists\, u\in H_0^1(\Omega)$ sol. de (P) :

$$(a(x)+|u|^q)\,|\nabla u|\in L^1(\Omega)\,,\ \ b(x)\,|u|^p\,|\nabla u|^2\in L^1(\Omega)\,,$$

$$\int_\Omega (a(x)+|u|^q)\,\nabla u\nabla\varphi+\int_\Omega b(x)\,|u|^{p-1}u\,|\nabla u|^2\,\varphi=\int_\Omega f\,\varphi\,,\ \ \forall\varphi\in W_0^{1,\infty}(\Omega)\,.$$

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Si $\mathbf{p}\,,\mathbf{q} > \mathbf{0} \Longrightarrow \exists\, u \in H^1_0(\Omega)$ sol. de (P) :

$$(a(x)+|u|^q)\,|\nabla u| \in L^1(\Omega)\,,\;\; b(x)\,|u|^p\,|\nabla u|^2 \in L^1(\Omega)\,,$$

$$\int_\Omega (a(x)+|u|^q)\,\nabla u\nabla\varphi + \int_\Omega b(x)\,|u|^{p-1}u\,|\nabla u|^2\,\varphi = \int_\Omega f\,\varphi\,,\;\; \forall\varphi\in W^{1,\infty}_0(\Omega)\,.$$

Ideas de la prueba. Dato acotado.

Caso 1 : $f \in L^{\infty}(\Omega)$

$$\begin{cases} -\mathrm{div}[\,(a(x) + |u|^{q})\ \nabla u\,] + b(x)\, u\, |u|^{p-1}\, |\nabla u|^{2} = \mathbf{f(x)}\,, & x \in \Omega\,, \\ u(x) = 0\,, & x \in \partial\Omega\,. \end{cases} \qquad \textbf{(P)}$$

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$$A_n(u) = -\text{div}\,[(a(x) + |T_n(u)|^q)\,\nabla u] + b(x)\ T_n(u)\,|T_n(u)|^{p-1}\,|\nabla u|^2$$

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- [Boccardo, Murat, Puel, 1992] $\Longrightarrow \exists u_n \in H_0^1(\Omega)\cap L^\infty(\Omega)$

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Paso al límite

$\exists\, u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ sol. de (P)

Ideas de la prueba. Caso general

$$\begin{cases} -\mathrm{div}[\,(a(x)+|u|^q)\ \nabla u\,] + b(x)\,u\,|u|^{p-1}\,|\nabla u|^2 = f(x)\,, & x\in\Omega\,, \\ u(x)=0\,, & x\in\partial\Omega\,. \end{cases} \qquad (P)$$

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Problemas aproximados

$$\begin{cases} -\mathrm{div}[\,(a(x)+|u_n|^q)\ \nabla u_n\,] + b(x)\,u_n\,|u_n|^{p-1}\,|\nabla u_n|^2 = f_n(x)\,, & x \in \Omega\,, \\ u(x) = 0\,, & x \in \partial\Omega\,, \end{cases} \qquad (P_n)$$

donde $f_n = T_n(f) \in L^\infty(\Omega)$.

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«Queremos pasar al límite......»

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- $\times\ T_k(u_n) \Longrightarrow \exists\, R > 0 \ : \ \|u_n\|_{H_0^1(\Omega)} \leq R$

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- $\times\ \dfrac{T_k(u_n)}{k} \Longrightarrow$

$$\int_\Omega (a(x) + |u_n|^q)\, |\nabla T_k(u_n)|^2 \leq \sqrt{k}\, \|f\|_{L^1(\Omega)} + k \int_{\{|u_n| > \sqrt{k}\}} |f|\,, \quad \forall k > 0$$

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- $\times\ T_1(u_n) \Longrightarrow \exists\, M > 0 \;:\; \displaystyle\int_\Omega b(x)|u_n|^p|\nabla u_n|^2 \leq M$
- $\times\ T_h(u_n - T_k(u)) \Longrightarrow \{\nabla u_n(x)\}$ converge hacia $\nabla u(x)$ a.e. en Ω

Cotas para u

- Para a.e. $k > 0$, u verifica

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$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, |u|^{p-1} u\, |\nabla u|^2\, \varphi = \int_\Omega f\, \varphi\,, \ \forall \varphi \in W_0^{1,\infty}(\Omega)\,.$$

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$$\int_\Omega (a(x) + |u|^q)\,\nabla u \nabla\varphi + \int_\Omega b(x)\,|u|^{p-1}u\,|\nabla u|^2\,\varphi \leq \int_\Omega f\,\varphi\,,\ \forall\varphi \in W_0^{1,\infty}(\Omega)\,.$$

« Lema de Fatou + Condición de signo »

Ideas de la prueba

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla\varphi + \int_\Omega b(x)\, |u|^{p-1}\, u|\nabla u|^2\varphi \le \int_\Omega f\,\varphi\,, \quad \forall\varphi \in W_0^{1,\infty}(\Omega)\,.$$

Ideas de la prueba

$$\int_\Omega (a(x) + |u|^q)\,\nabla u \nabla \varphi + \int_\Omega b(x)\,|u|^{p-1}\,u|\nabla u|^2 \varphi \,\le \int_\Omega f\,\varphi\,, \quad \forall \varphi \in W_0^{1,\infty}(\Omega)\,.$$

$$\int_\Omega (a(x) + |u|^q)\,\nabla u \nabla \varphi^+ + \int_\Omega b(x)\,|u|^{p-1}\,u|\nabla u|^2 \varphi^+ \le \int_\Omega f\,\varphi^+\,,\ \forall \varphi \in W_0^{1,\infty}(\Omega)$$

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, |u|^{p-1}\, u |\nabla u|^2 \varphi \, \le \int_\Omega f\, \varphi\,, \quad \forall \varphi \in W_0^{1,\infty}(\Omega)\,.$$

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$$+$$

Ideas de la prueba

$$\int_\Omega (a(x)+|u|^q)\,\nabla u \nabla \varphi + \int_\Omega b(x)\,|u|^{p-1}\,u|\nabla u|^2 \varphi \,\le \int_\Omega f\,\varphi\,,\quad \forall \varphi \in W_0^{1,\infty}(\Omega)\,.$$

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$$+$$

$$\int_\Omega (a(x)+|u|^q)\,\nabla u \nabla (-\varphi^-) + \int_\Omega b(x)|u|^{p-1}\,u|\nabla u|^2 (-\varphi^-) \le \int_\Omega f(-\varphi^-)\,, \forall \varphi \in W_0^{1,\infty}(\Omega)$$

$$\int_\Omega (a(x)+|u|^q)\nabla u\nabla\varphi + \int_\Omega b(x)\,|u|^{p-1}\,u|\nabla u|^2\varphi \leq \int_\Omega f\,\varphi\,, \quad \forall\varphi\in W_0^{1,\infty}(\Omega)\,.$$

Objetivo

$$\int_\Omega (a(x)+|u|^q)\nabla u\nabla\varphi^+ + \int_\Omega b(x)\,|u|^{p-1}\,u|\nabla u|^2\varphi^+ \leq \int_\Omega f\,\varphi^+\,, \ \forall\varphi\in W_0^{1,\infty}(\Omega)$$

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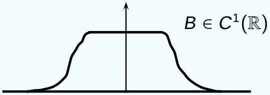
$$\int_\Omega (a(x)+|u|^q)\nabla u\nabla(-\varphi^-) + \int_\Omega b(x)|u|^{p-1}\,u|\nabla u|^2(-\varphi^-) \leq \int_\Omega f(-\varphi^-)\,, \forall\varphi\in W_0^{1,\infty}(\Omega)$$

Ideas de la prueba



$$B(s) = \begin{cases} 1 & \text{if } |s| \leq \frac{1}{2} \\ \in [0,1] & \text{if } \frac{1}{2} \leq |s| \leq 1 \\ 0 & \text{if } |s| \geq 1 \end{cases}$$

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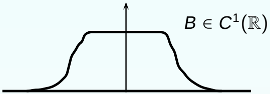

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$$H(s) = \frac{s\,|s|^p}{p+1}\,, \quad \forall s \in \mathbb{R}\,.$$

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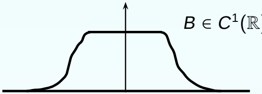

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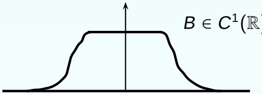
Dividimos la prueba en dos pasos

Ideas de la prueba



$$B(s) = \begin{cases} 1 & \text{if } |s| \leq \frac{1}{2} \\ \in [0,1] & \text{if } \frac{1}{2} \leq |s| \leq 1 \\ 0 & \text{if } |s| \geq 1 \end{cases}$$

$$H(s) = \frac{s\,|s|^p}{p+1}, \quad \forall s \in \mathbb{R}.$$

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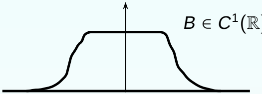
Fijamos $k > 0$ y $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ con $\psi \geq 0$



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Fijamos $k > 0$ y $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ con $\psi \ge 0$

$$\times \ \psi\, e^{\frac{-\nu H(u_n^-)}{\alpha}} B\left(\frac{u_n}{k}\right)$$

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$$\leq \|B'\|_\infty\, \|\psi\|_\infty \left(\frac{1}{\sqrt{k}} \|f\|_{L^1(\Omega)} + \int_{\{|u_n| > \sqrt{k}\}} |f|\right).$$

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$$+\int_\Omega b(x)\,|\nabla u_n|^2\,|u_n|^{p-1}\,u_n\,\psi\, e^{\frac{-\nu H(u_n^-)}{\alpha}}\, B\left(\frac{u_n}{k}\right) \le$$
$$\|B'\|_\infty\,\|\psi\|_\infty\left(\frac{1}{\sqrt{k}}\,\|f\|_{L^1(\Omega)}+\int_{\{|u_n|>\sqrt{k}\}}|f|\right).$$

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Límite cuando $n \to \infty$?

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$$\int_\Omega (a(x) + |u|^q)\ \nabla u \nabla \psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\ B\left(\frac{u}{k}\right)$$

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$\dfrac{\nu}{\alpha} a(x) \geq b(x)$ + Lema de Fatou

Ideas de la prueba

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$\forall\, \psi \in H_0^1(\Omega) \cap L^\infty(\Omega)\,,\ \psi \geq 0.$

Ideas de la prueba

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Paso II

Elegir ψ y pasar al límite cuando $k \to \infty$

Ideas de la prueba

$$\int_\Omega (a(x)+|u|^q)\,\nabla u \nabla\psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\, B\left(\frac{u}{k}\right)$$
$$-\frac{\nu}{\alpha}\int_\Omega (a(x)+|u|^q)\,\nabla u \nabla u^-\,|u^-|^p\,\psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\, B\left(\frac{u}{k}\right)$$
$$+\int_\Omega b(x)\,|\nabla u|^2\,|u|^{p-1}\,u\,\psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\, B\left(\frac{u}{k}\right)\le$$
$$\int_\Omega f\,\psi\, e^{\frac{-\nu H(u^-)}{\alpha}}\, B\left(\frac{u}{k}\right)+\|B'\|_\infty\,\|\psi\|_\infty\left(\frac{1}{\sqrt{k}}\,\|f\|_{L^1(\Omega)}+\int_{\{|u|>\sqrt{k}\}}|f|\right)$$

Paso II

Elegir ψ y pasar al límite cuando $k\to\infty$

$$\psi = e^{\frac{\nu H(u^-)}{\alpha}}\, B\left(\frac{u}{\sigma(k)}\right)\varphi^+\,,\quad \varphi\in W_0^{1,\infty}(\Omega)\,,$$

donde $\sigma(k)\longrightarrow+\infty$, cuando $k\to+\infty$.

Paso I + Paso II

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi^+ + \int_\Omega b(x)\, |u|^{p-1}\, u |\nabla u|^2 \varphi^+ \leq \int_\Omega f\, \varphi^+\,, \;\; \forall \varphi \in W_0^{1,\infty}(\Omega)$$

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De forma similar

Ideas de la prueba

Paso I + Paso II

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi^+ + \int_\Omega b(x)\, |u|^{p-1}\, u |\nabla u|^2 \varphi^+ \leq \int_\Omega f \varphi^+ , \;\; \forall \varphi \in W_0^{1,\infty}(\Omega)$$

De forma similar

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla (-\varphi^-) + \int_\Omega b(x) |u|^{p-1}\, u |\nabla u|^2 (-\varphi^-) \leq \int_\Omega f(-\varphi^-) , \forall \varphi \in W_0^{1,\infty}(\Omega)$$

$$\int_\Omega (a(x) + |u|^q)\, \nabla u \nabla \varphi + \int_\Omega b(x)\, |u|^{p-1} u\, |\nabla u|^2\, \varphi = \int_\Omega f \varphi , \;\; \forall \varphi \in W_0^{1,\infty}(\Omega) .$$

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