Ondas viajeras singulares en ecuaciones no lineales de reacción-difusión

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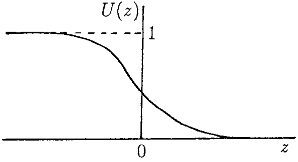
Classical traveling waves

Its is widely known that the model

$$u_t = \nu u_{xx} + k\,u(1-u)$$

displays classical traveling waves $u(t,x) = u(x - \sigma t)$ for wavespeeds $\sigma \geq 2\sqrt{k\nu}$. (Fisher; Kolmogoroff, Petrovsky, Piscounoff, 1937)



(Murray, *Mathematical Biology*)

The corresponding profiles are monotone. These solutions are supported in the whole line (as a subsidiary effect, information is propagated instantaneously).

Finite speed of propagation

Exponential tails can compromise some applications of this model. Can we fix that?

This would amount to use a diffusion term with the property of finite speed of propagation.

We will consider the following:

- the porous media equation
- flux saturation: the “relativistic heat equation”.

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Traveling waves for the porous media equation

The porous media equation (density-dependent diffusion):

$$u_t = \nu(u^{m-1}u_x)_x, \qquad m > 1.$$

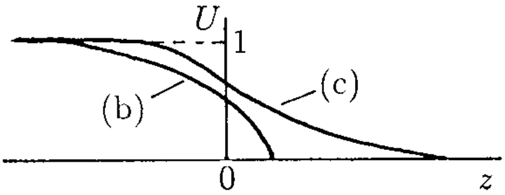
- Finite propagation speed.
- The heat equation is recovered for $m = 1$.

Traveling waves for the porous media equation

The following model admits traveling wave solutions:

$$u_t = \nu(u^{m-1}u_x)_x + k\,u(1-u).$$

(Newman, *J. Theor. Biol.* 1980)



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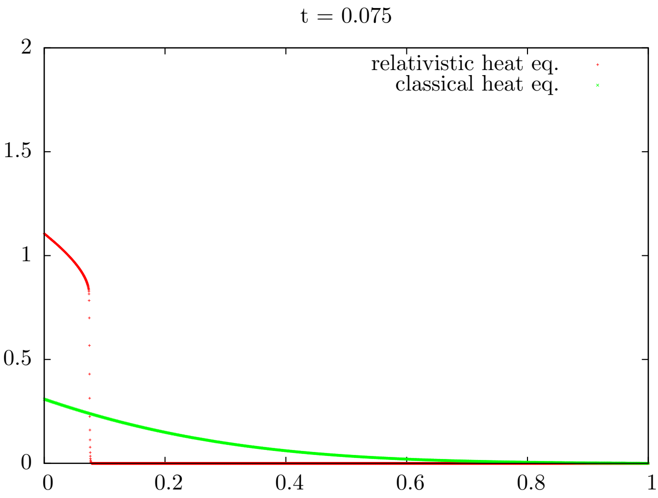
The “relativistic heat equation”: a numerical illustration



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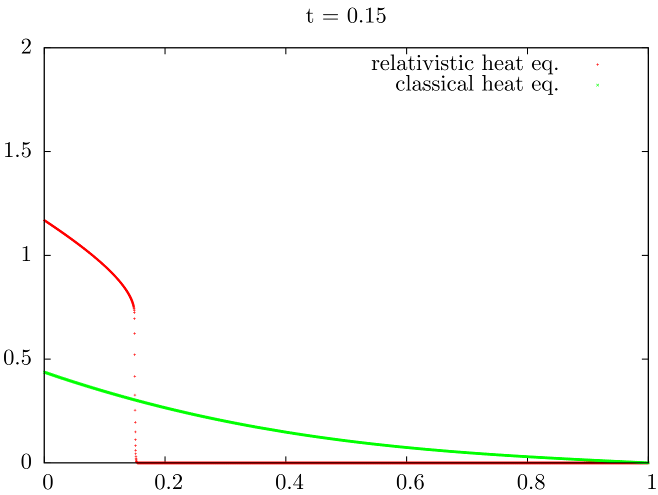

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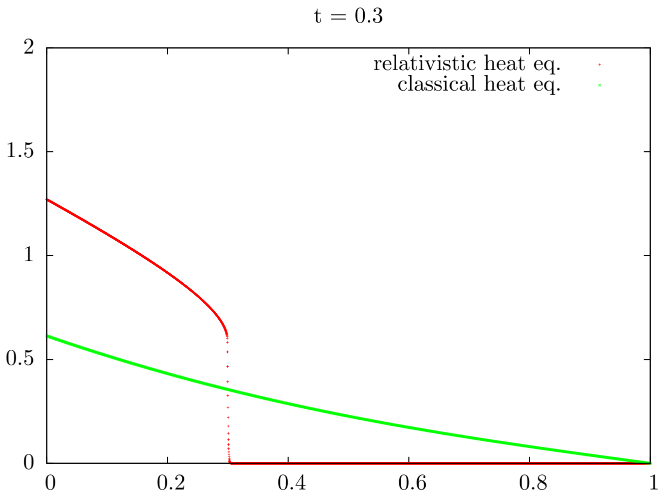

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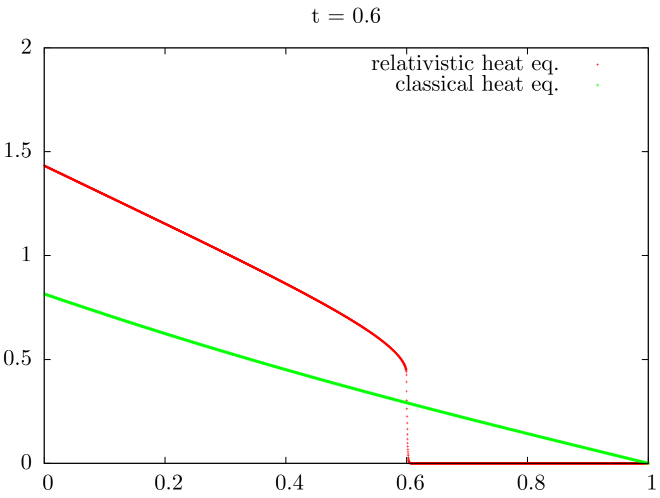

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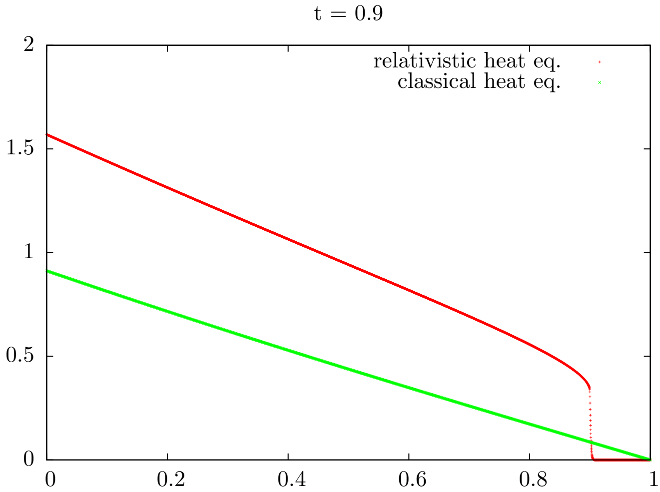

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The relativistic heat equation

$$\frac{\partial u}{\partial t} = \nu \operatorname{div} \left(\frac{|u|\nabla_x u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla_x u|^2}} \right)$$

- c maximum speed of propagation allowed.
- ν has the dimensions of a diffusion coefficient.

(Rosenau, Brennier)

The heat equation is embodied as a limit case ($c \to \infty$).

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Well-posedness: Difficulties of the analysis

- Very low spatial regularity available (handled in BV-type spaces).
- The meaning of the flux is not at all clear for non-smooth functions.
- Almost no time regularity expected.
- Uniqueness does not hold without extra requirements (entropy conditions).

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Our model problem

Now we want to mix the mechanisms of flux limitation and porous-media-type diffusion to try to produce new families of traveling waves (hopefully supported on a half-line). Thus, we consider the following equation for $m \geq 1$:

$$u_t = \nu \left(\frac{u^m u_x}{\sqrt{|u|^2 + \frac{\nu^2}{c^2}|u_x|^2}} \right)_x + k\, u(1-u).$$

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The traveling wave ansatz

We want solutions connecting the two constant states one and zero having a constant shape:

$$u(t,x) = u(\tau) = u(x - \sigma t), \quad \sigma > 0.$$

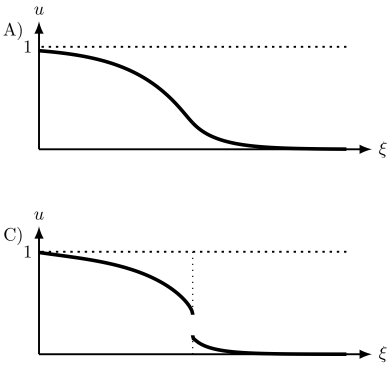
The profiles we are seeking are non-negative functions $u(\tau)$ defined on $]-\infty, \tau_\infty[$ for some $\tau_\infty \in]-\infty, +\infty]$, such that

$$\lim_{\tau \to -\infty} u(\tau) = 1$$

and

$$u'(\tau) < 0 \quad \forall \tau \in]-\infty, \tau_\infty[.$$

Catalog of solutions



(Campos, Guerrero, Sánchez, Soler, Ann. IHP 2013)

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Analytical results

Theorem (case $m = 1$, no porous media interactions)

Let $m = 1$. Consider the set

$$EQ(\sigma; c, \nu, k) = \left\{ r \in (0,1) : \frac{c^2}{\nu k}\left(\frac{\sigma}{c} - r\right) = \frac{\sqrt{1-r^2}}{r} \right\}, \ \sigma \geq 0.$$

The following results are verified:

- if EQ is not empty, then there exists a classical traveling wave solution. This is always the case for $\sigma > c$.
- if EQ is empty and $\sigma = c$ there exists a discontinuous traveling wave solution, supported in a half line.
- if EQ is empty and $\sigma \neq c$ then there are no traveling wave solutions.
- after suitable normalization, there is at most one traveling wave with speed σ.

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Let $m > 1$. The following results are verified:

i) There exist two values $0 < \sigma_{ent} < \sigma_{smooth} < mc$, depending on c, ν, m and k, such that:
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Reduction to a planar system

The traveling profile must solve the following equation:

$$\nu \left(\frac{u^m u'}{\sqrt{u^2 + \frac{\nu^2}{c^2}|u'|^2}} \right)' + \sigma u' + ku(1-u) = 0.$$

We set

$$r(\tau) = -\frac{\nu}{c} \frac{u'(\tau)}{\sqrt{|u(\tau)|^2 + \frac{\nu^2}{c^2}|u'(\tau)|^2}}.$$

Note: $r = 1$ iff $u' = -\infty$.

Thus, the second order ODE is equivalent to a first order planar dynamical system on a unit square:

$$\begin{cases} u' = & -\frac{c}{\nu}\frac{ru}{\sqrt{1-r^2}}, \\ r' = & \frac{1}{u^{m-1}}\frac{r}{\sqrt{1-r^2}}\left(mu^{m-1}\frac{c}{\nu}r - \frac{\sigma}{\nu}\right) + \frac{ku(1-u)}{cu^m}. \end{cases}$$

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The regime $\sigma \leq mc$, existence of barriers

When $m = 1$, the existence of solutions r^* to

$$\frac{c^2}{\nu k}\left(\frac{\sigma}{c} - r\right) = \frac{\sqrt{1-r^2}}{r}, \quad r \in]0,1[$$

allows to construct invariant sets of the form $]0,1[\times]0,r^*[$.

Thus, if there are equilibrium points then the associated traveling waves are classical.

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In this regime, orbits starting from the unstable manifold have a chance to reach the upper edge of the planar domain.

Whenever this takes place, the profile develops an infinite tangent. Is it possible to get a reasonable solution out of this?

Recall that our profiles are monotonically decreasing. A way out of the problem is to try to construct a profile with a downward jump discontinuity.

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Distributional solutions to our model having a jump discontinuity must satisfy the Rankine–Hugoniot jump condition:

$$V = \frac{F(u)^+ - F(u)^-}{u^+ - u^-}$$

- V velocity at which the jump discontinuity moves
- $u^\pm$ values of the solution at both sides of the discontinuity
- $F(u)^\pm$ values of the flux at both sides of the discontinuity

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Traveling waves that qualify as entropic solutions must satisfy this condition and must have infinite slopes at both sides of the discontinuity.

This means that we can extend our singular profile using a new orbit starting from the set $]0, 1[\times\{r = 1\}$ or the zero state.

About uniqueness

A uniqueness statement for solutions having the special form $(t,x) \mapsto u(x-\sigma t)$ holds in the following class:

- u is an entropy solution of the reaction-diffusion equation,
- u has its range in $[0,1]$ and is not the state zero nor the state one,
- there is a finite set of point p_i such that u is smooth in $\mathbb{R}\backslash\{p_1,\dots,p_n\}$.

Namely, given any value $\sigma \geq 0$, there is at most one solution of the form $(t,x) \mapsto u(x-\sigma t)$ in the former class (modulo spatial shifts). Note that no monotonicity assumptions on the profile are made: there are no traveling structures but the families of traveling waves that we have constructed.

Recap