Averaged alternating reflections in geodesic spaces

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An overview to the problem

Convex Feasibility Problem:

- $C_1, \ldots, C_N$ closed convex subsets of H, a Hilbert space.
- $C = \bigcap_{i=1}^{N} C_i \neq \emptyset$.

Find some point x in C.

▶ One frequently employed approach in solving the convex feasibility problem is algorithmic.

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Alternating Projection Method: von Neumann (1933)

- Given $x_1 \in H$, $x_{2n} = P_A(x_{2n-1})$, $x_{2n+1} = P_B(x_{2n})$, $n \in \mathbb{N}$.

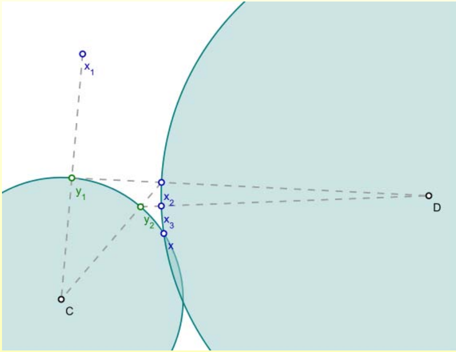

Figure: **APM**: Alternating Projection Method

Alternating Projection Method: von Neumann (1933)

It is known that

- A, B closed subspaces in $H \Rightarrow \{x_n\}$ converges in norm to a point in $A \cap B$.
- A, B closed convex sets in H with $A \cap B \neq \emptyset \Rightarrow \{x_n\}$ weakly converges to a point in $A \cap B$.

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Averaged Alternating Reflection Method

Another class of algorithms considered to solve the Convex feasibility problem bases on reflections instead of projections.

Reflection mapping

$x \in H$, $A, B \subseteq H$ nonempty, closed and convex.

- the reflection of a point x with respect to A is the image of x by the reflection mapping $R_A = 2P_A - I$.

- $T : H \to H$ defined as $T = \frac{R_A R_B + I}{2}$ (*NON-EXPANSIVE*).
- The *averaged alternating reflection method*, **AAR** , $x_0 \in H$ and $x_n = T^n x_0$ for every $n \in \mathbb{N}$.

- $\{x_n\}$ weakly converges to a fixed point of the mapping T and the projection of this point onto the set B lies in $A \cap B$.

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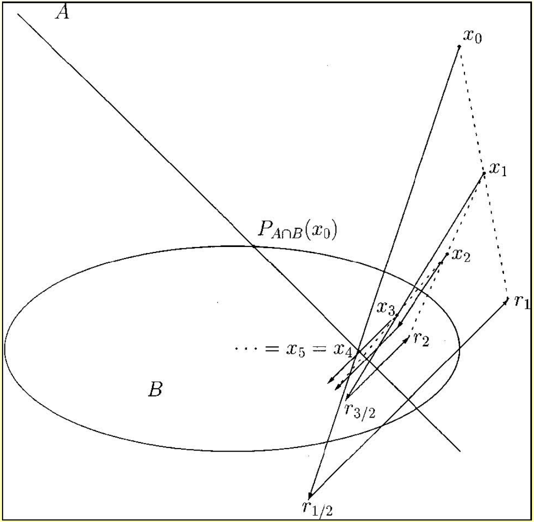

10th International Conference on Fixed Point Theory and its Applications (Cluj-Napoca, 2012)

Encouraging Problems

Conjecture 1 Reflections in spaces of constant curvature are nonexpansive.
Conjecture 2 Reflections in CAT(0) spaces are nonexpansive.

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Let (X, d) be a metric space.

- X is said to be a ***(uniquely) geodesic metric space*** if $\forall\, x, y \in X\ \exists$ a (unique) geodesic that joins them, i.e, a map

$$c : [0, l] \subseteq \mathbb{R} \to X \;:\; c(0) = x,\; c(l) = y \text{ and}$$

$$d(c(t), c(t')) = |t - t'|\ \forall\, t, t' \in [0, l].$$

- In this setting, $c : \mathbb{R} \to X$ such that $d(c(t), c(t')) = |t - t'|\ \forall\, t, t' \in \mathbb{R}$ is called a geodesic line.

Example: Any Banach space is a geodesic metric space with usual segments as geodesic segments.

Let X be a uniquely geodesic metric space and $[x, y]$ the unique geodesic segment between x and y.

- $A \subseteq X$ is said to be ***convex*** if $[x, y] \subset A$ for every $x, y \in A$.

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Model spaces

- The Euclidean space (curvature 0)
- The Spherical space (curvature 1)
- The Hyperbolic space (curvature -1)

The Spherical space

n-dimensional Sphere

The n-dimensional sphere $\mathbb{S}^n$ is the set of points $\{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid (x|x) = 1\}$, where $(\cdot|\cdot)$ denote the Euclidean scalar product.

Definition of the Spherical metric

Let $d : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ be the function that assigns to each pair of points A and B in the sphere the unique real number dist$(A, B) \in [0, \pi]$ such that $\mathbf{cos(d(A, B)) = (A|B)}$.

- This new function, the Spherical distance, is a metric.

Spherical space

$(\mathbb{S}^n, d)$ is called <u>Spherical space</u>.

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The Spherical space

Proposition

The Spherical space $(\mathbb{S}^n, d)$ is a geodesic metric space.

Spherical segment

Let:

- $a \in [0, \pi]$
- A a point in $(\mathbb{S}^n, d)$
- u a unit vector such that $(u|A) = 0$
  - A Spherical segment which join A and $c(a)$ will be the image of the interval $[0, a]$ by the geodesic c defined by $c(t) = (\cos t)A + (\sin t)u$.

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The Spherical space

Spherical triangle

Spherical triangle $\triangle$ in $\mathbb{S}^n$:

- Three different points p,q, and r in $\mathbb{S}^n$ (vertices)
- Three Spherical segments joining them pairwise.

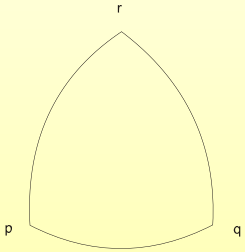

Figure: Spherical triangle

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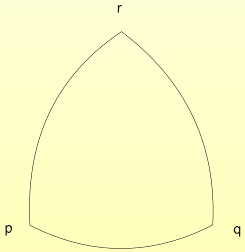

Figure: Spherical triangle

The Hyperbolic space

- $E^{n,1}$: vector space $\mathbb{R}^{n+1}$ endowed with the symmetric bilinear form that associates to vector u and v the real number

$$\langle u|v\rangle = -u_{n+1}v_{n+1} + \sum_{i=1}^{n} u_i v_i.$$

The upper sheet of the real hyperboloid

The upper sheet of the real hyperboloid, denoted by $\mathbb{H}^n$, is the set of points $\{u = (u_1, \ldots, u_{n+1}) \in E^{n,1} | \langle u|u\rangle = -1 \text{ and } u_{n+1} > 0\}$.

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The Hyperbolic space

- Hyperbolic metric.
- unique non-negative number $\text{dist}(A, B) \geq 0$ such that $\cosh \mathbf{d}(\mathbf{A}, \mathbf{B}) = -\langle \mathbf{A} | \mathbf{B} \rangle$.
- d : hyperbolic distance.
- $(\mathbb{H}^n, d)$ will be called the hyperbolic space.

Proposition

The Hyperbolic space $(\mathbb{H}^n, d)$ is a geodesic metric space.

- $A \in (\mathbb{H}^n, d)$, $u \in A^{\perp}$ a unit vector
- $c(t) = (\cosh t)A + (\sinh t)u$: hyperbolic geodesic.

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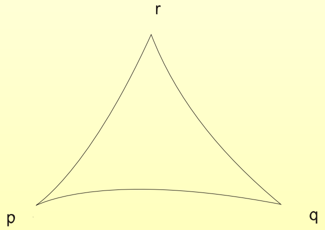

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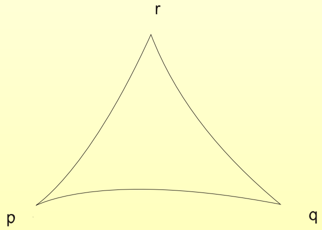

Figure: Hyperbolic triangle

The model spaces M_k^n

Let k be a real number.

Model spaces M_k^n

(1) If $k = 0$, M_0^n is the Euclidean space $\mathbb{E}^n$;

(2) If $k > 0$, M_k^n is obtained from the Spherical space $\mathbb{S}^n$ by multiplying the distance function by $1/\sqrt{k}$;

(3) If $k < 0$, M_k^n is obtained from the Hyperbolic space $\mathbb{S}^n$ by multiplying the distance function by $1/\sqrt{-k}$.

- $\mathbb{E}^n = M_0^n$,
- $\mathbb{S}^n = M_1^n$,
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CAT(k) spaces

Conjectures

CAT(k) inequality. Comparison axiom

- (M, d) metric space.
- k real number.
- $\triangle$ geodesic triangle in M which perimeter is less than $2D_k$, where D_k denotes the diameter of M_k^n: $D_k = \pi/\sqrt{k}$ if $k > 0$, $D_k = \infty$ if $k \leq 0$.
- $\overline{\triangle} \subseteq M_k^2$ a comparison triangle for $\triangle$.

▶ $\triangle$ satisfy the **CAT(k) inequality** if:

$x, y \in \triangle$
$\bar{x}, \bar{y} \in \overline{\triangle}$

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

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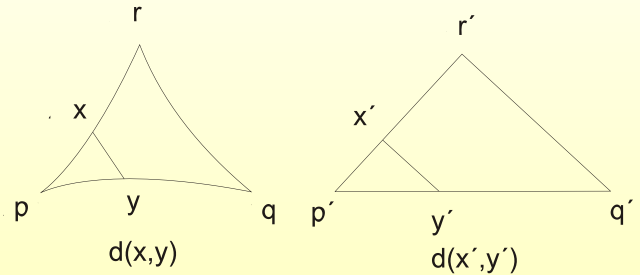

Figure: CAT(k) inequality

CAT(k) spaces

CAT(k) space

- M is a CAT(k) space for $k \leq 0$ if:
  - M is a geodesic space.
  - All its geodesic triangles satisfy the CAT(k) inequality.
- M is a CAT(k) space for $k > 0$ if:
  - M is D_k-geodesic.
  - All geodesic triangles in M of perimeter less than $2D_k$ satisfy the CAT(k) inequality.

Contents

How to deal with the non-linear case

- The metric projection onto C is well-defined and single-valued.
- (X, d) uniq. geodesic with the geodesic extension prop.
- The reflection of $x \in X$ w.r.t C can be any z in a geodesic line $\gamma \supset [x, P_C x]$ for which $P_C x = \frac{x+z}{2}$.
- If X has no bifurcating geodesics, then geodesics can be extended in a unique way (z unique: $R_C x$).
- Another problem consists in guaranteeing certain properties of T (Hilbert: P_C and R_C are nonexpansive and consequently T is firmly nonexpansive).
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- $x \in X$
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Conjecture 2 Reflections in CAT(0) spaces are nonexpansive.

Proposition

- *C a closed convex subset of M_k^n.*
- *$x, y \in M_k^n$ such that $d(x, C), d(y, C) < D_k/2$.*
- *Then*
$$d(R_C x, R_C y) \le d(x, y).$$

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K. Goebel and S. Reich , Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, *Pure and Applied Mathematics, Marcel Dekker*, Inc. New York and Basel, 1984.

- **Example** 22.1: the reflection mapping in the (complex) Hilbert ball is not nonexpansive.

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P.L Lions and B. Mercier , Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, 16, (1979) 964-979.

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Theorem

- A and B two nonempty closed convex subsets of a **Hilbert** space H.
- $A \cap B \neq \emptyset$.
- $x_0 \in H$ and x_n the sequence starting at x_0 generated by the AAR method.

- $\{x_n\}_{n \geq 1}$ **weakly converges** to some fixed point of the mapping T and $P_B x \in A \cap B$.
- The "shadow" sequence $\{P_B x_n\}_{n \geq 1}$ is **bounded** and each of its **weak cluster points** belongs to $A \cap B$.

P.L Lions and B. Mercier , Splitting algorithms for the sum of two nonlinear operators, *SIAM J. Numer. Anal.*, 16, (1979) 964-979.

Theorem

- A and B two nonempty closed convex subsets of M_k^n for $k \leq 0$.
- $A \cap B \neq \emptyset$.
- $x_0 \in M_k^2$ and x_n the sequence starting at x_0 generated by the AAR method.
- $\{x_n\}_{n\geq 1}$ **converges** to some fixed point of the mapping T and $P_B x \in A \cap B$.
- The “shadow” sequence $\{P_B x_n\}_{n\geq 1}$ is **convergent** and its limit belongs to $A \cap B$.

- **Important fact**: $T = \frac{R_A R_B + I}{2}$ is nonexpansive since a CAT(0) space is Busemann convex.

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Theorem

- *A, B and C nonempty closed convex subsets of M^n_k for $k > 0$.*
- *$A, B \subseteq C$, diam$(C) < D_k/2$ and $R_B(C), R_A(C) \subseteq C$.*
- *$x_0 \in M^2_k$ and x_n the sequence starting at x_0 generated by the AAR method.*

- *Any **convergent subsequence** $\{x_{n_k}\}$ of $\{x_n\}$ **converges** to some fixed point of the mapping T and $P_B x \in A \cap B \neq \emptyset$.*
- *The "shadow" sequence $\{P_B x_n\}_{n \geq 1}$ is **bounded** and each of its **cluster points** belongs to $A \cap B$.*

Gracias por su atención