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On the smoothness of L^p of a positive vector measure

L. Agud, **J.M. Calabuig**, E.A. Sánchez Pérez

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Outline

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Motivation of the work...

A norm attaining operator $T : X \to Y$ is...

a bounded linear map between two Banach spaces X and Y satisfying that there is $0 \neq x \in X$ such that $\|T(x)\|_Y = \|T\| \cdot \|x\|_X$.

A Banach space X is said to be smooth if...

for every $0 \neq x \in X$ there exists a unique $x^* \in X^*$ norming x, i.e.,

$$\|x^*\| = 1 \qquad \text{and} \qquad \langle x, x^* \rangle = \|x\|.$$

In the case when X is smooth the unique norming element $x^* \in X^*$ for $0 \neq x \in X$ is denoted by $\Theta_X(x)$.

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Motivation of the work...

Theorem (Howard and Schep)

R. Howard and A.R. Schep, *Norms of positive operators on L^p-spaces*, Proceedings of the American Mathematical Society **109**(1) (1990), 135–146.

Let $T : X \to Y$ be a linear and bounded operator between smooth Banach spaces. Given $x \in X$, the following assertions are equivalent:

(a) *T attains its norm at x.*

(b) $T^*(\Theta_Y(T(x))) = \|T\| \cdot \Theta_X(x)$.

Actually in the paper only the implication (a)⇒(b) is shown; however for the converse it suffices to notice that

$$\begin{aligned} \|T(x)\|_Y &= \langle T(x), \Theta_Y(T(x))\rangle = \langle x, T^*(\Theta_Y(T(x)))\rangle \\ &= \langle x, \|T\|\Theta_X(x)\rangle = \|T\|\langle x, \Theta_X(x)\rangle = \|T\| \cdot \|x\|_X. \end{aligned}$$

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Motivation of the work...

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Example (Smooth spaces)

B. Beauzamy, *Introduction to Banach spaces and their geometry*, North-Holland, Amsterdam, 1982.

1. Given a scalar measure μ — even if μ a σ-finite measure — and $1 < p < \infty$ then $L^p(\mu)$ is smooth. Moreover for every $0 \neq f \in L^p(\mu)$ the unique norming element for f is given by

$$\Theta_{L^p(\mu)}(f) = \frac{sgn(f)|f|^{p-1}}{\|f\|_{L^p(\mu)}^{p-1}} \in L^p(\mu)^* = L^{p'}(\mu),$$

where, as usual, $1/p + 1/p' = 1$.

2. The spaces $L^1(\mu)$ and $L^\infty(\mu)$ —even in the case when μ is an atomic measure— are not smooth.

WHAT ABOUT THE CASE WHEN m IS A (POSITIVE) VECTOR MEASURE?

Outline

Integration of scalar functions w.r. to vector measures

Let (Ω, Σ) be a finite measure space.

X (real) Banach space, B_X (closed) unit ball, S_X sphere.

Let $m : \Sigma \to X$ be a (countable) additive vector measure.

A measurable function $f : \Omega \to \mathbb{R}$ is said to be integrable with respect to m if

1. f is integrable with respect to $\langle m, x^* \rangle$, for all $x^* \in X^*$, where

$$\langle m, x^* \rangle : \Sigma \to \mathbb{R}, \qquad \langle m, x^* \rangle(A) = \langle m(A), x^* \rangle.$$

2. For all $A \in \Sigma$ there is a unique $x_A \in X$ such that

$$\int_A f d\langle m, x^* \rangle = \langle x_A, x^* \rangle, \qquad x^* \in X^*.$$

The vector x_A is denoted by $\int_A fdm$.

Integration of scalar functions w.r. to vector measures

$L^1(m)$

Set consisting of classes of functions identifying functions that are equal m-a.e.

Is an order continuous Banach function space with the norm

$$\|f\|_{L^1(m)} = \sup\{\int_\Omega |f| d\big|\langle m, x^*\rangle\big| : x^* \in B_{X^*}\},$$

and over any Rybakov measure,that is, a scalar measure of the form $\nu = \big|\langle m, x^*\rangle\big|$ where $x^* \in X^*$ being equivalent to m.

In the case when X is a Banach lattice $m : \Sigma \to X$ is said to be positive if $m(A) \geq 0$ for all $A \in \Sigma$. In such case it is well-known that

$$\|f\|_{L^1(m)} = \Big\| \int_\Omega |f| dm \Big\|_X.$$

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Integration of scalar functions w.r. to vector measures

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$L^p(m)$

$$1 < p < \infty, \qquad f \in L^p(m) \text{ if and only if } |f|^p \in L^1(m).$$

Example

For $1 \leq p < \infty$ the space $L^p(m)$ fails to be smooth.

Problem

When is smooth the space $L^p(m)$ for $1 < p < \infty$ and m positive?

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The key lemma

For a fixed $x_0^* \in S_{X^*}$ let $\nu = \langle m, x_0^* \rangle$ be the associated Rybakov measure.

The topological dual $L^p(m)^*$ coincides with its Köthe dual $L^p(m)'$, which is

$$L^p(m)' = \{h \ \ \Sigma\text{-measurable} : fh \in L^1(\nu) \text{ for all } f \in L^p(m)\},$$

and the duality is given by the formula

$$\langle h, f \rangle = \int_\Omega fh d\nu.$$

Let us consider now the integration operator $I_m : L^1(m) \to X$ associated to a vector measure m, and defined by $I_m(f) = \int_\Omega fdm$, for $f \in L^1(m)$.

By using Radon-Nikodým derivatives $I_m^* : X^* \to L^1(m)^* = L^1(m)'$ can be written as

$$I_m^*(x^*)(f) = \int_\Omega f \frac{d\langle m, x^* \rangle}{d\nu} d\nu, \quad x^* \in X^*, f \in L^1(m).$$

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The key lemma

Consider the subspace of $L^1(m)'$ given by

$$\mathbf{R}(m) = \{I_m^*(x^*) = \frac{d\langle m, x^*\rangle}{d\nu} : x^* \in B_{X^*}\} \subseteq B_{L^1(m)'}.$$

Given $1 < p < \infty$, the pointwise product space $B_{L^{p'}(m)} \cdot \mathbf{R}(m)$ is defined as

$$B_{L^{p'}(m)} \cdot \mathbf{R}(m) = \{h \in L^0(\nu) : h = g \cdot I_m^*(x^*), g \in B_{L^{p'}(m)}, x^* \in B_{X^*}\}.$$

It is easy to see that

$$B_{L^{p'}(m)} \cdot \mathbf{R}(m) \subseteq B_{L^p(m)'}.$$

The key lemma

Lemma

Let $m : \Sigma \to X$ be a positive vector measure and $1 < p < \infty$. The following assertions are equivalent:

(a) $B_{L^{p'}(m)} \cdot \mathbf{R}(m)$ *is convex and closed.*

(b) $B_{L^p(m)'} \subseteq B_{L^{p'}(m)} \cdot \mathbf{R}(m)$.

The weak topology on L^p of a vector measure. I. Ferrando and J. Rodríguez, Top. Appl. **155**(13) (2008), 1439–1444.

Outline

The results

Theorem

Let $1 < p < \infty$ *and* $m : \Sigma \to X$ *be a positive vector measure satisfying:*

(i) $B_{L^{p'}(m)} \cdot \mathbf{R}(m)$ *is convex and closed.*

(ii) X *is smooth.*

Then $L^p(m)$ *is smooth.*

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Sketch of the proof

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(1) Let $f \in S_{L^p(m)}$. Since m is positive then

$$\|f\|_{L^p(m)}^p = \Big\| \int_\Omega |f|^p dm \Big\|_X.$$

Therefore, by using the Hanh-Banach Theorem we get that there is $x_f^* \in B_{X^*}$ (that we can assume $x_f^* \geq 0$) such that

$$\langle \int_\Omega |f|^p dm, x_f^* \rangle = 1. \tag{1}$$

(2) Define now the linear map $\varphi : L^p(m) \to \mathbb{R}$ given by

$$\varphi(h) = \int_\Omega h g_f \, d\langle m, x_f^* \rangle, \qquad h \in L^p(m). \tag{2}$$

where $g_f = sgn(f)|f|^{p-1} \in S_{L^{p'}(m)}$.
Claim 1. $\varphi \in S_{L^p(m)'}$ and it norms f.

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Sketch of the proof

(3) Assume $\psi \in S_{L^p(m)'}$ and also norms f. Hipothesis (i) together with the lemma give $g \in B_{L^{p'}(m)}$ and $x^* \in B_{X^*}$ such that

$$\psi = g \cdot \frac{d\langle m, x^* \rangle}{d\nu}. \tag{3}$$

Define now the positive measure $\eta : \Sigma \to [0, \infty[$ defined by

$$\eta(A) = \frac{d\langle m, x^* \rangle}{d\nu} \cdot \nu(A) \text{ for all } A \in \Sigma.$$

Claim 2. $f \in S_{L^p(\eta)}$, $g \in S_{L^{p'}(\eta)}$ and g norms f (as a function on $L^p(\eta)$).

(4) Since $L^p(\eta)$ is smooth and $sgn(f)|f|^{p-1} = g_f$ also norms f in $L^p(\eta)$ then

$$g = sgn(f)|f|^{p-1} = g_f \text{ in } L^p(\eta). \tag{4}$$

(5) *Claim 3.* x_f^* and x^* norm $x = \int_\Omega |f|^p dm \in S_X$.
Therefore the smoothness of X gives $x = x_f^*$.

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Sketch of the proof

(6) The result is then consequence of:

$$\begin{aligned} \psi(h) &= \int_\Omega hg \frac{d\langle m, x^*\rangle}{d\nu} d\nu = \int_\Omega hg d\eta = \int_\Omega hg_f d\eta \\ &= \int_\Omega hg_f \frac{d\langle m, x^*\rangle}{d\nu} d\nu = \int_\Omega hg_f d\langle m, x_f^*\rangle = \varphi(h). \end{aligned}$$

Theorem

Let $1 < p < \infty$ and $m : \Sigma \to X$ be a positive vector measure satisfying:

(i) $B_{L^{p'}(m)} \cdot \mathbf{R}(m)$ *is convex and closed.*

(ii) $L^1(m)$ *is smooth.*

Then $L^p(m)$ is smooth.

Some examples

The following examples show cases where the inclusion

$$B_{L^p(m)'} \subseteq B_{L^{p'}(m)} \cdot \mathbf{R}(m) \tag{5}$$

holds.

Some examples

Example (A first easy one)

Let (Ω, Σ, μ) be a positive finite measure space and $1 < p < \infty$ and consider the vector measure $m_0 : \Sigma \to L^1(\mu)$ given by $m_0(A) = \chi_A$ for each $A \in \Sigma$.
Then it is well-known that

$L^p(m_0)$ is isometrically isomorphic to $L^p(\mu)$.

Take the Rybakov measure associated to the function $\chi_\Omega \in L^\infty(\mu)$. Note that in such case $\nu = \langle m_0, \chi_\Omega \rangle = \mu$, since

$$\nu(A) = \langle m_0, \chi_\Omega \rangle (A) = \langle m_0(A), \chi_\Omega \rangle = \int_A \chi_\Omega \, d\mu = \mu(A), \qquad A \in \Sigma. \tag{6}$$

For this vector measure, the relation that appears in formula (5) is

$$B_{L^{p'}(\mu)} \subseteq B_{L^{p'}(\mu)} \cdot \mathbf{R}(m_0).$$

But this containment is trivially satisfied considering the decomposition $g = g\chi_\Omega$ for each $g \in L^{p'}(\mu)$.

Some examples

Example (Smoothness)

The space

$$X = \bigoplus_4 L^2(\mu_{|A_i}),$$

where μ is the Lebesgue measure on $[0,1]$ and $(A_i)_{i\geq 1}$ is a disjoint family of measurable sets of $[0,1]$ is smooth.

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