Espacios de Banach de funciones continuas

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Theorem (Banach-Stone)

If $C(K)$ and $C(L)$ are isometric, then K is homeomorphic to L.

What happens when $C(K)$ and $C(L)$ are just isomorphic? That is, when there is an operator $T : C(K) \longrightarrow C(L)$ whose inverse $T^{-1} : C(L) \longrightarrow C(K)$ is also an operator.

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In case 1, $C(K)$ has separable dual. In case 2, $C(K)$ has nonseparable dual.

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In the separable case, it is known when X^* is nonseparable (Rosenthal) and some other particular cases (Bourgain).

In the nonseparable case the problem remains open as well.

Injective spaces

Definition

A Banach space E is injective if every operator $T : X \longrightarrow E$ can be extended to any superspace $Y \supset X$.

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If E is injective, must E be isomorphic to a $C(K)$ space? indeed, to a 1-injective $C(K)$?

Few operators

Theorem (Gowers-Maurey)

There exists a Banach space X such that every operator $T : X \longrightarrow X$ is of the form $T = \lambda \cdot I + S$ where $\lambda \in \mathbb{R}$ and S is strictly singular.

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Such an X cannot be a $C(K)$ space, because we always have multiplication operators: If $f \in C(K)$, we have the operator $f \cdot I : C(K) \longrightarrow C(K)$.

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1. $C(K)$ is indecomposable.
2. $C(K)$ is not isomorphic to any $C(L)$ with L 0-dimensional.
3. $C(K)$ is not isomorphic to its hyperplanes.
4. K does not contain convergent sequences.
5. Every continuous function $\varphi : K \longrightarrow K$ is either the identity or constant.

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$C(K)$ is not isomorphic to any $C(L)$ with L 0-dimensional.

Boolean $C(K)$

Only two examples are known of $C(K)$ which is not isomorphic to $C(L)$ with L 0-dimensional: The aformentioned by Koszmider and Avilés-Koszmider.

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Let B be the closed ball of a nonspearable Hilbert space in its weak topology. Is $C(B)$ isomorphic to $C(L)$ with L 0-dimensional?