Shell interactions for Dirac operators

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Free Dirac operator in $\mathbb{R}^3$

$H : \mathcal{C}_c^\infty(\mathbb{R}^3)^4 \to \mathcal{C}_c^\infty(\mathbb{R}^3)^4$ free Dirac operator in $\mathbb{R}^3$,

$$H = -i\alpha\cdot\nabla + m\beta = \begin{pmatrix} m & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & m & \partial_1 + i\partial_2 & -\partial_3 \\ \partial_3 & \partial_1 - i\partial_2 & -m & 0 \\ \partial_1 + i\partial_2 & -\partial_3 & 0 & -m \end{pmatrix}.$$

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REMARKS:

- 1st order symmetric differential operator.
- Local version of $\sqrt{-\Delta + m^2}$: $H^2 = (-\Delta + m^2)I_4$.
- Introduced by Dirac (1928) to study the electron (*Quantum Physics*).

Coupling with a singular potential

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MOTIVATION:

- *Quantum Physics* requires self-adjointness.
- Previous results on $H + \lambda\delta_{|x|=1}$ (and other V's on S^2).
- $\delta_{|x|=1} \sim \frac{1}{|x|}$ under scaling & $\frac{1}{|x|}$ critical for H.

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$(H + V)(\varphi) \in L^2(\mathbb{R}^3)^4 \Longrightarrow (H + V)(\varphi) = G$ for some $G \in L^2(\mathbb{R}^3)^4$.

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Therefore $\quad \varphi = \phi * (G + g) \quad$ **and**

$$(H + V)(\varphi) = G, \qquad V(\varphi) = -g,$$

where ϕ is the fundamental solution of $H = -i\alpha \cdot \nabla + m\beta$,

$$\phi(x) = \frac{e^{-m|x|}}{4\pi|x|}\left(m\beta + (1 + m|x|)\, i\alpha \cdot \frac{x}{|x|^2}\right) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}.$$

Some properties of $\phi * (G + g)$

$\Omega_+ \subset \mathbb{R}^3$ bounded regular domain, $\quad \Omega_- = \mathbb{R}^3 \setminus \overline{\Omega_+}, \quad \Sigma = \partial\Omega_\pm,$
σ surface measure on Σ, $\quad N$ normal vector on Σ w.r.t. Ω_+.

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If $g \in L^2(\sigma)^4$, then $H(\phi * g) = 0$ in Σ^c.
For $x \in \Sigma$, set

$$C_\pm g(x) = \lim_{\Omega_\pm \ni y \overset{nt}{\to} x} (\phi * g)(y), \quad C_\sigma g(x) = p.v.(\phi * g)(x).$$

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Then,

- $C_\pm = \mp \frac{i}{2} (\alpha \cdot N) + C_\sigma$ (*Plemelj-Sokhotski jump formulae*),
- $\big(C_\sigma(\alpha \cdot N)\big)^2 = -\frac{1}{4}$.

Abstract theorem

THEOREM:

Given $\Lambda : L^2(\sigma)^4 \to L^2(\sigma)^4$ bounded, self-adjoint and with closed range, define

$$D = \{\phi * (G + g) : (\phi * G)|_\Sigma = \Lambda(g)\}.$$

If $V(\phi * (G + g)) = -g$,
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REMARKS:

- Under more assumptions on Λ, $H + V$ is self-adjoint.
- Other symmetric differential operators.
- Other singular measures.
- Other relations between $(\phi * G)|_\Sigma$ and g.

THEOREM:

Given $\lambda \in \mathbb{R}$, define

$$D = \left\{\varphi = \phi * (G + g) : \lambda(\phi * G)|_\Sigma = -(1 + \lambda C_\sigma)(g)\right\}.$$

If $V_\lambda(\varphi) = \frac{\lambda}{2}(\varphi_+ + \varphi_-)$ ($\varphi_\pm$ n.t. boundary values of φ on Σ),
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There exist $\{\lambda_j\}_{j\in\mathbb{N}} \subset (0, \infty)$ accumulating at 2 such that

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- and if $|\lambda| = \lambda_j$ for some j, there exists $0 \neq \varphi \in D$ so that

$$(H+V_\lambda)(\varphi) = 0 \quad \text{or} \quad (H+V_{-\lambda})(\varphi) = 0.$$

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Moreover, if $\big\{C_\sigma(\alpha\cdot N)\,,\,\Lambda(\alpha\cdot N)\big\} = -\big(\Lambda(\alpha\cdot N)\big)^2$,
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APPLICATION:

$$V_{a,b}(\varphi) = \tfrac{a}{2}(\varphi_+ + \varphi_-) + \tfrac{b}{2}\beta(\varphi_+ + \varphi_-), \quad a, b \in \mathbb{R}.$$

$H + V_{a,b}$ makes Σ impenetrable if $a^2 - b^2 + 4 = 0$.

The end.