Weighted inequalities for the strong maximal function

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Main Ingredients of this talk

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- Conditions on the weights: Muckenhoupt weights, **bump** conditions, **tauberian** conditions.

1. Maximal functions and weighted norm inequalities.

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2. Two-weight problem

Contents

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- The collection $\mathfrak{R}$ consists of all rectangles in $\mathbb{R}^n$ with sides parallel to the coordinate axes.
- These two bases are quite different: The *Hardy-Littlewood maximal function* M is a one-parameter maximal function while *the strong maximal function* M_s is an n-parameter maximal operator.

- The Hardy-Littlewood maximal theorem says that

$$|\{Mf > \lambda\}| \lesssim_n \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda}, \quad \lambda > 0,$$
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$$M_s f \leq M_1 \circ \cdots \circ M_n f.$$

By the Hardy-Littlewood maximal theorem we get the L^p-bounds ($1 < p \leq \infty$) for the strong maximal function.

Boundedness properties

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- A subtle use of $M_s f(x) \leq M_1 \circ \cdots \circ M_n f$ can also give the endpoint distribution inequality

$$|\{M_s f > \lambda\}| \lesssim_n \int_{\mathbb{R}^n} \frac{|f|}{\lambda}\left(1 + \left(\log^+ \frac{|f|}{\lambda}\right)^{n-1}\right), \quad \lambda > 0$$

B. Jessen, J. Marcinkiewicz & A. Zygmund (1935).

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  - This covering argument defines *good* rectangles that are still *good* rectangles in one dimension less.
  - This kind of induction argument enphasizes (AGAIN) the one-dimensional nature of this operator.

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$$[w]_{A_p,\mathfrak{B}} := \sup_{B\in\mathfrak{B}} \left(\frac{1}{|B|}\int_B w\right)\left(\frac{1}{|B|}\int_B w^{1-p'}\right)^{p-1} < +\infty, \quad 1 < p < +\infty$$

$$[w]_{A_1,\mathfrak{B}} := \inf\{c > 0 : M_{\mathfrak{B}} w \le cw \text{ almost everywhere}\} < +\infty.$$

The classes $A_{p,\mathfrak{B}}$ are increasing with p. We set $A_{\infty,\mathfrak{B}} := \bigcup_{p>1} A_{p,\mathfrak{B}}$.

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- For simplicity set $A_p := A_{p,\mathfrak{Q}}$ and $A_p^* := A_{p,\mathfrak{R}}$ = "strong A_p-weights."

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- $w \in A_p^*$ if and only if $w(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \in A_p$ uniformly in $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$.

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Two weight problem

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- This problem is already solve for a general basis $\mathcal{B}$ in terms of a **testing condition**.

$$\int_G M_{\mathfrak{B}}(\mathbf{1}_G \sigma)^p u \leq c\sigma(G),$$

where $\sigma = v^{1-p'}$ and $M_{\mathcal{B}}^{\sigma} : L^p(\sigma) \to L^p(\sigma)$.

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- These kind of conditions involves the operator itself.

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GOAL: to obtain **sufficient** condition **SIMILAR** in form to the A_p condition.

First attempt for M

Condition $A_{p,r}$

$$\sup_Q \left(\frac{1}{|Q|}\int_Q u\,dx\right)\left(\frac{1}{|Q|}\int_Q v^{r(1-p')}\,dx\right)^{\frac{p-1}{r}} < \infty, \quad r > 1.$$

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and we can finish using the Marcinkiewicz interpolation theorem.

Renormalizing the problem

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Bump conditions II

- Generalization: Orlicz bump condition

Orlicz norm

Given a Young function (continous, convex, strictly increasing) B we can define the following *localized* norm:

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$$\|w\|_{L^p,Q} \leq \|w\|_{L^p(\log L)^\alpha,Q} \leq c\,\|w\|_{L^{pr},Q} \qquad r > 1, \alpha > 0$$

Bump condition III

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THANK YOU VERY MUCH!!