Multilinear local Tb Theorem for Square functions

Ana Grau de la Herrán – University of Helsinki

(joint work with J. Hart and L. Oliveira)

Goal

\[ \| Tf(x) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \]
Goal

\[ \| Tf(x) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \]

Examples

Singular integral operators

\[ Tf(x) = \int K(x, y) f(y) \, dy \]

Riesz Transform

\[ R_j f(x) = \int |x_j - y_j|^{n+1} \frac{f(y)}{|x - y|} \, dy \]
Goal

\[ \| Tf(x) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \]

Examples

- Singular integral operators

\[ Tf(x) = \int K(x, y)f(y)dy \]
Goal

$$\|Tf(x)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

Examples

- **Singular integral operators**
  $$Tf(x) = \int K(x, y)f(y)\,dy$$

- **Riesz Transform**
  $$R_j f(x) = \int \frac{|x_j - y_j|}{|x - y|^{n+1}} f(y)\,dy$$
Goal

\[ \|Tf(x)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \]

Examples

- Singular integral operators

\[ Tf(x) = \int K(x, y)f(y)dy \]
Goal

\[ \| Tf(x) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \]

Examples

- Singular integral operators

\[ Tf(x) = \int K(x, y)f(y)dy \]

- Hilbert Transform

\[ Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{x - y}dy \]
Goal

\[ \|Tf(x)\|_{L^P(\mathbb{R}^n)} \leq C\|f\|_{L^P(\mathbb{R}^n)} \]

Examples

- Singular integral operators

\[ Tf(x) = \int K(x, y)f(y)dy \]
Goal

\[ \| Tf(x) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)} \]

Examples

- Singular integral operators
  
  \[ Tf(x) = \int K(x, y) f(y) dy \]

- Square functions
  
  \[ Sf(x) = \left( \int_0^\infty \left( \int \psi_t(x, y) f(y) dy \right)^2 dt / t \right)^{1/2} \]
Goal

\[ \|Tf(x)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \]

Examples

- **Singular integral operators**
  \[ Tf(x) = \int K(x, y)f(y)dy \]

- **Square functions**
  \[ Sf(x) = \left( \int_0^\infty \left| \int \Psi_t(x, y)f(y)dy \right|^2 dt/t \right)^{1/2} \]

where we note \( \theta_t f(x) = \int \Psi_t(x, y)f(y)dy \).
Examples

- Kato problem \((AHLM_c T)\)

\[
\| \sqrt{L} f \|_2 \approx \| \nabla f \|_2
\]
Examples

- Kato problem \((AHLMcT)\)

\[
\| \sqrt{L} f \|_2 \approx \| \nabla f \|_2
\]

We reduce ourselves to prove the \(L^2\) boundedness problem for the Square function

\[
\theta_t = -\frac{1}{(1 + t^2L)^{-1}} t \text{div} A
\]
Examples

- Kato problem \((AHLM_c T)\)

\[ \| \sqrt{L} f \|_2 \approx \| \nabla f \|_2 \]

We reduce ourselves to prove the \(L^2\) boundedness problem for the Square function

\[ \theta_t = -(1 + t^2 L)^{-1} t \text{div} A \]

- Single layer potential \((GH,R)\)

\[ \theta_t = t (\partial_t)^2 S_t \]

where \(S_t f(x) = \int E(x, t, y, 0) f(y) dy\).
We have set a problem
• We have set a problem
• We have some operators as motivation
We have set a problem
We have some operators as motivation
How do we solve the problem?
We have set a problem
We have some operators as motivation
How do we solve the problem?

Method

1. To solve the problem for $p = 2$.
2. Tool = To test how the operator behaves locally.
   - Constant function
   - An accretive function $b$
   - Locally adapted functions.
We have set a problem
We have some operators as motivation
How do we solve the problem?

Method
1. To solve the problem for \( p = 2 \).
- We have set a problem
- We have some operators as motivation
- How do we solve the problem?

**Method**

1. To solve the problem for $p = 2$.
2. Tool = To test how the operator behave locally.
We have set a problem
- We have some operators as motivation
- How do we solve the problem?

Method

1. To solve the problem for $p = 2$.
2. Tool = To test how the operator behave locally.
   - Constant function 1.
- We have set a problem
- We have some operators as motivation
- How do we solve the problem?

**Method**

1. To solve the problem for \( p = 2 \).
2. Tool = To test how the operator behave locally.
   - Constant function 1.
   - An accretive function \( b \).
We have set a problem
We have some operators as motivation
How do we solve the problem?

Method

1. To solve the problem for \( p = 2 \).
2. Tool = To test how the operator behave locally.
   - Constant function 1.
   - An accretive function b.
   - Locally adapted functions.
Let \( \theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y)f(y)dy \), and assume that \( \psi_t \) is a LP family. Suppose that we have the Carleson measure estimate

\[
\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\theta_t 1(x)|^2 \frac{dxdt}{t} \leq C
\]

Then we have that the square function is bounded in \( L^2 \).

**Remark.**
The converse direction is essentially due to Fefferman and Stein.
Tb Theorem for Square functions [Semmes]

Let $\theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy$, assume that $\psi_t$ is a LP family and $b$ an accretive function.
Suppose that we have the Carleson measure estimate

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q \left| \int_Q \theta_t b(x) \right|^2 \frac{dxdt}{t} \leq C$$

Then we have the $L^2$ boundedness of the square function.

**Remark.**
An accretive function $b$ is an $L^\infty$ function that satisfies $\Re(b) \geq c > 0$.
Assume that $\theta_t$ is a Calderón Zygmund operator and $1 < p < \infty$.

Suppose also that there exist $\delta > 0$, $C_0 < \infty$ such that for any dyadic cube $Q$, there exists a function $b_Q$ satisfying:

(i) $\int_{\mathbb{R}^n} |b_Q|^p \leq C_0 |Q|$

(ii) $\int_Q \left( \int_0^{\ell(Q)} |\theta_t b_Q(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C_0 |Q|$

(iii) $\delta \leq \left| f_Q b_Q \right|$

Then we have $L^2$ boundedness of the square function.
Formal definition of the operators

Definition (scalar-valued kernel)

Given a function \( \Psi_t: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C} \) we define the square function associated to the function \( \Psi_t \):

\[
S(f)(x) := \left( \int_0^\infty \left| \int_{\mathbb{R}^n} \Psi_t(x, y) f(y) \, dy \right|^2 \, dt \right)^{1/2}
\]

where \( f: \mathbb{R}^n \rightarrow \mathbb{C} \).

Our goal is,

\[
\left( \int_{\mathbb{R}^n} |S(f)(x)|^2 \, dx \right)^{1/2} \leq C \|f\|_2
\]
Formal definition of the operators

Definition (escalar-valued kernel)

Given a function $\Psi_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ we define the **Square function** associated to the function $\Psi_t$:

$$
S(f)(x) := \left( \int_0^{\infty} \left| \int_{\mathbb{R}^n} \Psi_t(x, y)f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}
$$

where $f : \mathbb{R}^n \to \mathbb{C}$.
Definition (escalar-valued kernel)

Given a function $\Psi_t : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ we define the **Square function** associated to the function $\Psi_t$:

$$S(f)(x) := \left( \int_0^\infty \left| \int_{\mathbb{R}^n} \Psi_t(x, y)f(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{C}$.

Our goal is,

$$\left( \int_{\mathbb{R}^n} |S(f)(x)|^2 dx \right)^{1/2} \leq C\|f\|_2$$
Definition (vector-valued kernel)

Given a function \( \Psi_t = (\Psi^1_t, \ldots, \Psi^m_t) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^m \) we define the **Square function** associated to the function \( \Psi_t \):

\[
S(f)(x) := \left( \int_0^\infty \left| \int_{\mathbb{R}^n} \Psi_t(x, y) \cdot f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2}
\]

\[
= \left( \int_0^\infty \left| \sum_{j=1}^m \int_{\mathbb{R}^n} \Psi^j_t(x, y) f_j(y) dy \right|^2 \frac{dt}{t} \right)^{1/2}
\]

where \( f : \mathbb{R}^n \to \mathbb{C}^m (f = (f_1, \ldots, f_m)) \).
Definition (vector-valued kernel)

Given a function $\Psi_t = (\Psi_t^1, \ldots, \Psi_t^m) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^m$ we define the Square function associated to the function $\Psi_t$:

$$S(f)(x) := \left( \int_0^\infty \left| \int_{\mathbb{R}^n} \Psi_t(x, y) \cdot f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2}$$

$$= \left( \int_0^\infty \left| \sum_{j=1}^m \int_{\mathbb{R}^n} \Psi_t^j(x, y)f_j(y) dy \right|^2 \frac{dt}{t} \right)^{1/2}$$

where $f : \mathbb{R}^n \to \mathbb{C}^m$ ($f = (f_1, \ldots, f_m)$).

Our goal is,

$$\left( \int_{\mathbb{R}^n} |S(f)(x)|^2 dx \right)^{1/2} \leq C\|f\|_2$$
Definition (multilinear square function)

Given a function $\Psi_t : \mathbb{R}^{(m+1)n} \to \mathbb{C}$ we define the Square function associated to the function $\Psi_t$:

$$S_t(f_1, ..., f_m)(x) := \int_{\mathbb{R}^{n \times m}} \Psi_t(x, y_1, ..., y_m)f_1(y_1)dy_1...f_m(y_m)dy_m$$

$$S(f_1, ..., f_m)(x) := \left( \int_{0}^{\infty} \left| S_t(f_1, ..., f_m)(x) \right|^2 \frac{dt}{t} \right)^{1/2}$$

where $f_1, ..., f_m : \mathbb{R}^n \to \mathbb{C}$. 
Definition (multilinear square function)

Given a function $\Psi_t : \mathbb{R}^{(m+1)n} \rightarrow \mathbb{C}$ we define the **Square function** associated to the function $\Psi_t$:

$$S_t(f_1, ..., f_m)(x) := \int_{\mathbb{R}^{n \times m}} \Psi_t(x, y_1, ..., y_m) f_1(y_1) dy_1 ... f_m(y_m) dy_m$$

$$S(f_1, ..., f_m)(x) := \left( \int_0^\infty |S_t(f_1, ..., f_m)(x)|^2 \frac{dt}{t} \right)^{1/2}$$

where $f_1, ..., f_m : \mathbb{R}^n \rightarrow \mathbb{C}$.

Our goal is,

$$\left( \int_{\mathbb{R}^n} |S(f_1, ..., f_m)(x)|^2 dx \right)^{1/2} \leq C \prod_{i=1}^m \|f_i\|_{p_i}, \quad 1/2 = \sum_{i=1}^m \frac{1}{p_i}.$$
Definition (multi-parameter square function)

Given a function $\Psi_t : \mathbb{R}^{n \times n} \rightarrow \mathbb{C}$ we define the Square function associated to the function $\Psi_t$:

$$S_t(f_1, ..., f_m)(x) := \int_{\mathbb{R}^n} \Psi_t(x, y_1, ..., y_m)f_1(y_1)dy_1...f_m(y_m)dy_m$$

$$S(f_1, ..., f_m)(x) := \left(\int_0^\infty |S_t(f_1, ..., f_m)(x)|^2 \frac{dt}{t}\right)^{1/2}$$

where $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{C}$ for $i = 1, ..., m$ and $n = \sum_{i=1}^m n_i$. 
Definition (multi-parameter square function)

Given a function $\Psi_t : \mathbb{R}^{n \times n} \rightarrow \mathbb{C}$ we define the **Square function** associated to the function $\Psi_t$:

$$S_t(f_1, ..., f_m)(x) := \int_{\mathbb{R}^n} \Psi_t(x, y_1, ..., y_m)f_1(y_1)dy_1...f_m(y_m)dy_m$$

$$S(f_1, ..., f_m)(x) := \left( \int_0^\infty |S_t(f_1, ..., f_m)(x)|^2 \frac{dt}{t} \right)^{1/2}$$

where $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{C}$ for $i = 1, ..., m$ and $n = \sum_{i=1}^m n_i$.

Our goal is,

$$\left( \int_{\mathbb{R}^n} |S(f_1, ..., f_m)(x)|^2 dx \right)^{1/2} \leq C \prod_{i=1}^m \|f_i\|_2.$$
We say that $\Psi_t : \mathbb{R}^{(m+1) \times n} \to \mathbb{C}$ is a multilinear standard Calderón-Zygmund kernel if for all $x, y_1, ..., y_m, x', y'_1, ..., y'_m \in \mathbb{R}^n$ satisfies:

1. $|\Psi_t(x, y_1, ..., y_m)| \leq C \frac{t^{-mn}}{\prod_{i=1}^{m} (1+t^{-1}|x-y_i|)^{n+\alpha}}$

2. $|\Psi_t(x, y_1, ..., y_m) - \Psi_t(x, y_1, ..., y'_i, ..., y_m)| \leq C \frac{t^{-mn}(t^{-1}|y_i-y'_i|)^{\alpha}}{\prod_{i=1}^{m} (1+t^{-1}|x-y_i|)^{n+\alpha}}$

3. $|\Psi_t(x, y_1, ..., y_m) - \Psi_t(x', y_1, ..., y_m)| \leq C \frac{t^{-mn}(t^{-1}|x-x'|)^{\alpha}}{\prod_{i=1}^{m} (1+t^{-1}|x-y_i|)^{n+\alpha}}$
Comparison Standard C-Z kernel with Multilinear C-Z kernel

(Size condition)

\[ |\Psi_t(x, y)| \leq C \frac{t^\alpha}{(t + |x - y|)^{n+\alpha}} \]

vs

\[ |\Psi_t(x, y_1, \ldots, y_m)| \leq C \frac{t^{\alpha m}}{\prod_{i=1}^{m}(t + |x - y_i|)^{n+\alpha}} \]
Comparison Standard C-Z kernel with Multilinear C-Z kernel

\[ |\Psi_t(x, y) - \Psi_t(x, y')| \leq C\frac{|y - y'|^\alpha}{(t + |x - y|)^{n+\alpha}} \]

\[ \text{vs} \]

\[ |\Psi_t(x, y_1, ..., y_m) - \Psi_t(x, y_1, ..., y'_i, ..., y_m)| \leq C\frac{|y_i - y'_i|^\alpha}{\prod_{i=1}^m(t + |x - y_i|)^{n+\alpha}} \]
Comparison Standard C-Z kernel with Multilinear C-Z kernel

(Hölder condition on $x$)

\[ |\Psi_t(x, y) - \Psi_t(x', y)| \leq C \frac{|x - x'|^\alpha}{(t + |x - y|)^{n+\alpha}} \]

\[ \text{vs} \]

\[ |\Psi_t(x, y_1, ..., y_m) - \Psi_t(x', y_1, ..., y_m)| \leq C \frac{|x - x'|^\alpha}{\prod_{i=1}^{m} (t + |x - y_i|)^{n+\alpha}} \]
Let $S$ be a multilinear square function associated to $\Psi_t$ a multilinear standard C-Z kernel. Suppose that there exists $q_i, q > 1$ for $i = 1, \ldots, m$ with $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$ and functions $b^i_Q$ indexed by dyadic cubes $Q \subset \mathbb{R}^n$ for $i = 1, \ldots, m$ such that for every dyadic cube there exists $C > 0$ such that

1. $\int_{\mathbb{R}^n} |b^i_Q|^{q_i} \leq B_1 |Q|$
2. $\frac{1}{B_2} \leq \left| \frac{1}{|Q|} \int_Q \prod_{i=1}^{m} b^i_Q(x) dx \right|$
3. $\left| \frac{1}{|R|} \int_R \prod_{i=1}^{m} b^i_Q(x) dx \right| \leq B_3 \prod_{i=1}^{m} \left| \frac{1}{|R|} \int_R b^i_Q(x) dx \right|$ for all dyadic subcubes $R \subset Q$
4. $\int_Q \left( \int_{\ell(Q)} |S_t(b^1_Q, \ldots, b^m_Q)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \leq B_4 |Q|$

Then

$\left( \int_{\mathbb{R}^n} |S(f_1, \ldots, f_m)(x)|^2 dx \right)^{1/2} \leq C \prod_{i=1}^{m} \|f_i\|_p, \quad 1/2 = \sum_{i=1}^{m} \frac{1}{p_i}$. 
Remark 1.- We say that \{b_Q\} is a pseudo-accretive system if it satisfies the first two conditions.

Remark 2.- We say that \{b_Q\} is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.
Remark 1.- We say that \( \{b_Q\} \) is a pseudo-accretive system if it satisfies the first two conditions.

Remark 2.- We say that \( \{b_Q\} \) is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.

Examples

- m-compatible system

\[ \epsilon \leq b_j Q(x) \leq \epsilon - \frac{1}{2} \]

Characteristic functions

\[ b_Q(x) = \chi_Q(x) \]

Gaussian functions

\[ b_Q(x) = e^{-|x - x_Q|^2 \ell(Q)^2} \]

Poisson kernels

\[ b_Q(x) = \ell(Q)^n + \frac{\ell(Q^2 + |x - x_Q|^2)}{2} \]

Non m-compatible system

\[ b_Q(x) = b_1 Q(x) = b_2 Q(x) = \left( x - \frac{1}{2} \right) \chi_{[0,2]}(x) \]
Remark 1.- We say that \( \{b_Q\} \) is a pseudo-accretive system if it satisfies the first two conditions.

Remark 2.- We say that \( \{b_Q\} \) is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.

Examples

- m-compatible system
  
  1. \( \epsilon \leq b_Q^j(x) \leq \epsilon^{-1} \)
Remark 1. - We say that \( \{b_Q\} \) is a pseudo-accretive system if it satisfies the first two conditions.

Remark 2. - We say that \( \{b_Q\} \) is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.

**Examples**

- **m-compatible system**
  1. \( \epsilon \leq b^j_Q(x) \leq \epsilon^{-1} \)
  2. Characteristic functions \( b_Q(x) = \chi_Q(x) \)
Remark 1.- We say that \( \{b_Q\} \) is a pseudo-accretive system if it satisfies the first two conditions.

Remark 2.- We say that \( \{b_Q\} \) is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.

**Examples**

- **m-compatible system**
  1. \( \epsilon \leq b^j_Q(x) \leq \epsilon^{-1} \)
  2. Characteristic functions \( b_Q(x) = \chi_Q(x) \)
  3. Gaussian functions \( b_Q(x) = e^{-\frac{|x-x_Q|^2}{\ell(Q)^2}} \)
Remark 1.- We say that \( \{b_Q\} \) is a pseudo-accretive system if it satisfies the first two conditions.

Remark 2.- We say that \( \{b_Q\} \) is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.

Examples

- **m-compatible system**
  1. \( \epsilon \leq b^j_Q(x) \leq \epsilon^{-1} \)
  2. Characteristic functions \( b_Q(x) = \chi_Q(x) \)
  3. Gaussian functions \( b_Q(x) = e^{\frac{-|x-x_Q|^2}{\ell(Q)^2}} \)
  4. Poisson kernels \( b_Q(x) = \frac{\ell(Q)^{n+1}}{(\ell(Q)^2 + |x-x_Q|^2)^{(n+1)/2}} \)
Remark 1.- We say that \( \{b_Q\} \) is a pseudo-accretive system if it satisfies the first two conditions.

Remark 2.- We say that \( \{b_Q\} \) is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.

Examples

- **m-compatible system**
  1. \( \epsilon \leq b^j_Q(x) \leq \epsilon^{-1} \)
  2. Characteristic functions \( b_Q(x) = \chi_Q(x) \)
  3. Gaussian functions \( b_Q(x) = e^{-\frac{|x-x_Q|^2}{\ell(Q)^2}} \)
  4. Poisson kernels \( b_Q(x) = \frac{\ell(Q)^{n+1}}{(\ell(Q)^2 + |x-x_Q|^2)^{(n+1)/2}} \)

- **non m-compatible system**

\[
b_Q(x) = b^1_Q(x) = b^2_Q(x) = (x - 1/2)\chi_{[0,2]}(x)
\]
Multilinear T1 Theorem (GLMY, GO and H)

Let $R$ be a Square function whose kernel satisfies the multilinear size condition and the multilinear Hölder condition on the $y$ variables. If $R_t(1, ..., 1) = 0$ for $t > 0$ then

$$\|R(f_1, ..., f_m)\|_{L^p} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}}$$

Remark.-
The condition $R_t(1, ..., 1) = 0$ is a sufficient but not necessary condition. We also have a version on the T1 theorem where the requirement is that we have the Carleson measure

$$\sup_Q \int_0^{\ell(Q)} \int_Q |S_t(1, ..., 1)|^2 \frac{dxdt}{t} \leq C$$
Proposition

For every dyadic cube $Q$ there exists a family of subcubes $\{Q_k\}$, $C > 0$ and $\eta \in (0, 1)$ such that

1. $\sum_k |Q_k| < (1 - \eta)|Q|$
2. $\int_Q \left( \int_{\tau_Q(x)} \left| S_t(1, \ldots, 1) \right|^2 \frac{dt}{t} \right)^{q/2} \leq C|Q|$

where $E = Q \setminus \bigcup_k Q_k$ and $\tau_Q(x) = \begin{cases} \ell(Q_k) & x \in Q_k \\ 0 & x \in E \end{cases}$
Lemma

There exists $N > 0$ and $\beta \in (0, \infty)$ such that for every dyadic cube $Q$

$$|\{x \in Q : g_Q > N\}| \leq (1 - \beta)|Q|$$

where

$$g_Q(x) = \left( \int_0^{\ell(Q)} |S_t(1, \ldots, 1)(x)|^2 \frac{dt}{t} \right)^{1/2}$$
Muchas gracias!!!