Multilinear local Tb Theorem for Square functions

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(joint work with J. Hart and L. Oliveira)

September 19, 2013. Sevilla.

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$$||Tf(x)||_{L^p(\mathbb{R}^n)} \leq C||f||_{L^p(\mathbb{R}^n)}$$

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$$Tf(x) = \int K(x,y)f(y)dy$$

Riesz Transform

$$R_j f(x) = \int \frac{|x_j - y_j|}{|x-y|^{n+1}} f(y)dy$$

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Hilbert Transform

$$Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{x-y}dy$$

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$$Tf(x) = \int K(x,y)f(y)dy$$

- Square functions

$$Sf(x) = \left(\int_0^\infty \left| \int \Psi_t(x,y)f(y)dy \right|^2 dt/t \right)^{1/2}$$

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$$Sf(x) = \left(\int_0^\infty \left|\int \Psi_t(x,y)f(y)dy\right|^2 dt/t\right)^{1/2}$$

where we note $\theta_t f(x) = \int \Psi_t(x,y)f(y)dy$.

Examples

- Kato problem ($AHLM_cT$)

$$\|\sqrt{L}f\|_2 \approx \|\nabla f\|_2$$

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We reduce ourselves to prove the L^2 boundedness problem for the Square function

$$\theta_t = -(1 + t^2 L)^{-1} t div\ A$$

- Single layer potential (GH,R)

$$\theta_t = t(\partial_t)^2 S_t$$

where $S_t f(x) = \int E(x,t,y,0) f(y) dy$.

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Method

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2. Tool = To test how the operator behave locally.
   - Constant function 1.
   - An accretive function b.
   - Locally adapted functions.

T1 Theorem for Square functions [Christ-Journé]

Let $\theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x,y) f(y) dy$, and assume that ψ_t is a LP family. Suppose that we have the Carleson measure estimate

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\theta_t 1(x)|^2 \frac{dxdt}{t} \leq C$$

Then we have that the square function is bounded in L^2.

Remark.-

The converse direction is essentially due to Fefferman and Stein

Tb Theorem for Square functions [Semmes]

Let $\theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x,y) f(y) dy$, assume that ψ_t is a LP family and b an accretive function

Suppose that we have the Carleson measure estimate

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\theta_t b(x)|^2 \frac{dxdt}{t} \leq C$$

Then we have the L^2 boundedness of the square function.

Remark.-

An accretive function b is an L^∞ function that satisfies $\Re e(b) \geq c > 0$

Local Tb Theorem for Square functions [HMc, HLMc,AHLMCT]

Assume that θ_t is a Calderón Zygmund operator and $1 < p < \infty$.

Suppose also that there exist $\delta > 0$, $C_0 < \infty$ such that for any dyadic cube Q, there exists a function b_Q satisfying:

(i) $\int_{\mathbb{R}^n} |b_Q|^p \leq C_0|Q|$

(ii) $\int_Q \left(\int_0^{\ell(Q)} |\theta_t b_Q(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C_0|Q|$

(iii) $\delta \leq \left| \fint_Q b_Q \right|$

Then we have L^2 boundedness of the square function.

Formal definition of the operators

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Definition (escalar-valued kernel)

Given a function $\Psi_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ we define the **Square function** associated to the function Ψ_t:

$$S(f)(x) := \left(\int_0^\infty \left| \int_{\mathbb{R}^n} \Psi_t(x,y) f(y) dy \right|^2 \frac{dt}{t} \right)^{1/2}$$

where $f : \mathbb{R}^n \to \mathbb{C}$.

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Our goal is,

$$\left(\int_{\mathbb{R}^n} |S(f)(x)|^2 dx \right)^{1/2} \leq C \|f\|_2$$

Definition (vector-valued kernel)

Given a function $\Psi_t = (\Psi_t^1, ..., \Psi_t^m) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^m$ we define the **Square function** associated to the function Ψ_t:

$$S(f)(x) := \left(\int_0^\infty \left|\int_{\mathbb{R}^n} \Psi_t(x,y) \cdot f(y)dy\right|^2 \frac{dt}{t}\right)^{1/2}$$
$$= \left(\int_0^\infty \left|\sum_{j=1}^m \int_{\mathbb{R}^n} \Psi_t^j(x,y) f_j(y)dy\right|^2 \frac{dt}{t}\right)^{1/2}$$

where $f : \mathbb{R}^n \to \mathbb{C}^m$ $(f = (f_1, ..., f_m))$.

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Definition (multilinear square function)

Given a function $\Psi_t : \mathbb{R}^{(m+1)n} \to \mathbb{C}$ we define the **Square function** associated to the function Ψ_t:

$$S_t(f_1, ..., f_m)(x) := \int_{\mathbb{R}^{n\times m}} \Psi_t(x, y_1, ..., y_m) f_1(y_1) dy_1 ... f_m(y_m) dy_m$$

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$$\left(\int_{\mathbb{R}^n} |S(f_1, ..., f_m)(x)|^2 dx \right)^{1/2} \leq C \prod_{i=1}^{m} ||f_i||_{p_i}, \quad 1/2 = \sum_{i=1}^{m} \frac{1}{p_i}.$$

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Multilinear standard Calderón-Zygmund kernel

We say that $\Psi_t : \mathbb{R}^{(m+1)\times n} \to \mathbb{C}$ is a multilinear standard Calderón-Zygmund kernel if for all $x, y_1, ..., y_m, x', y_1', ..., y_m' \in \mathbb{R}^n$ satisfies:

1. $|\Psi_t(x, y_1, ..., y_m)| \leq C \frac{t^{-mn}}{\prod_{i=1}^m (1+t^{-1}|x-y_i|)^{n+\alpha}}$
2. $|\Psi_t(x, y_1, ..., y_m) - \Psi_t(x, y_1, ..., y_i', ..., y_m)| \leq C \frac{t^{-mn}(t^{-1}|y_i-y_i'|)^{\alpha}}{\prod_{i=1}^m (1+t^{-1}|x-y_i|)^{n+\alpha}}$
3. $|\Psi_t(x, y_1, ..., y_m) - \Psi_t(x', y_1, ..., y_m)| \leq C \frac{t^{-mn}(t^{-1}|x-x'|)^{\alpha}}{\prod_{i=1}^m (1+t^{-1}|x-y_i|)^{n+\alpha}}$

Comparison Standard C-Z kernel with Multilinear C-Z kernel

1. (Size condition)

$$|\Psi_t(x,y)| \leq C \frac{t^\alpha}{(t+|x-y|)^{n+\alpha}}$$

vs

$$|\Psi_t(x,y_1,...,y_m)| \leq C \frac{t^{\alpha m}}{\prod_{i=1}^m (t+|x-y_i|)^{n+\alpha}}$$

Comparison Standard C-Z kernel with Multilinear C-Z kernel

2. (Hölder condition on y)

$$|\Psi_t(x,y) - \Psi_t(x,y')| \leq C \frac{|y-y'|^\alpha}{(t+|x-y|)^{n+\alpha}}$$

vs

$$|\Psi_t(x,y_1,...,y_m) - \Psi_t(x,y_1,...,y_i',...,y_m)| \leq C \frac{|y_i-y_i'|^\alpha}{\prod_{i=1}^m (t+|x-y_i|)^{n+\alpha}}$$

Comparison Standard C-Z kernel with Multilinear C-Z kernel

3. (Hölder condition on x)

$$|\Psi_t(x,y) - \Psi_t(x',y)| \leq C\frac{|x-x'|^\alpha}{(t+|x-y|)^{n+\alpha}}$$

vs

$$|\Psi_t(x,y_1,...,y_m) - \Psi_t(x',y_1,...,y_m)| \leq C\frac{|x-x'|^\alpha}{\prod_{i=1}^m (t+|x-y_i|)^{n+\alpha}}$$

(Multilinear Tb Theorem)

Let S be a multilinear square function associated to Ψ_t a multilinear standar C-Z kernel. Suppose that there exists $q_i, q > 1$ for $i = 1, ..., m$ with $\frac{1}{q} = \sum_{i=1}^{m} \frac{1}{q_i}$ and functions b_Q^i indexed by dyadic cubes $Q \subset \mathbb{R}^n$ for $i = 1, ..., m$ such that for every dyadic cube there exists $C > 0$ such that

1. $\int_{\mathbb{R}^n} |b_Q^i|^{q_i} \leq B_1 |Q|$
2. $\frac{1}{B_2} \leq \left| \frac{1}{|Q|} \int_Q \prod_{i=1}^{m} b_Q^i(x) dx \right|$
3. $\left| \frac{1}{|R|} \int_R \prod_{i=1}^{m} b_Q^i(x) dx \right| \leq B_3 \prod_{i=1}^{m} \left| \frac{1}{|R|} \int_R b_Q^i(x) dx \right|$ for all dyadic subcubes $R \subset Q$
4. $\int_Q \left(\int_0^{\ell(Q)} |S_t(b_Q^1, ..., b_Q^m)(x)|^2 \frac{dt}{t} \right)^{\frac{q}{2}} dx \leq B_4 |Q|$

Then

$$\left(\int_{\mathbb{R}^n} |S(f_1, ..., f_m)(x)|^2 dx \right)^{1/2} \leq C \prod_{i=1}^{m} \|f_i\|_{p_i}, \quad 1/2 = \sum_{i=1}^{m} \frac{1}{p_i}.$$

Remark 1.- We say that $\{b_Q\}$ is a pseudo-accretive system if it satisfies the first two conditions.
Remark 2.- We say that $\{b_Q\}$ is a m-compatible collection of pseudo-accretive system if it satisfies the first three conditions.

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- non m-compatible system

$$b_Q(x) = b_Q^1(x) = b_Q^2(x) = (x - 1/2)\chi_{[0,2]}(x)$$

Multilinear T1 Theorem (GLMY, GO and H)

Let R be a Square function whose kernel satisfies the multilinear size condition and the multilinear Hölder condition on the y variables. If $R_t(1, ..., 1) = 0$ for $t > 0$ then

$$||R(f_1, ..., f_m)||_{L^p} \leq C \prod_{i=1}^{m} ||f||_{L^{p_i}}$$

Remark.-

The condition $R_t(1, ..., 1) = 0$ is a sufficient but not necessary condition. We also have a version on the T1 theorem where the requirement is that we have the Carleson measure $\sup_Q \int_0^{\ell(Q)} \int_Q |S_t(1, ..., 1)|^2 \frac{dxdt}{t} \leq C$

Proposition

For every dyadic cube Q there exists a family of subcubes $\{Q_k\}$, $C > 0$ and $\eta \in (0, 1)$ such that

- $\sum_k |Q_k| < (1 - \eta)|Q|$
- $\int_Q \left(\int_{\tau_Q(x)}^{\ell(Q)} |S_t(1, ..., 1)|^2 \frac{dt}{t} \right)^{q/2} \leq C|Q|$

where $E = Q \setminus \bigcup_k Q_k$ and $\tau_Q(x) = \begin{cases} \ell(Q_k) & x \in Q_k \\ 0 & x \in E \end{cases}$

Lemma

There exists $N > 0$ and $\beta \in (0, \infty)$ such that for every dyadic cube Q

$$|\{x \in Q : g_Q > N\}| \leq (1 - \beta)|Q|$$

where

$$g_Q(x) = \left(\int_0^{\ell(Q)} |S_t(1, ..., 1)(x)|^2 \frac{dt}{t} \right)^{1/2}$$

Muchas gracias!!!