Atomic blocks for martingales

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- $(H_1)^* = \mathrm{BMO}$.

Atoms

- A function a is called a **classical p-atom** $(1 < p \leq \infty)$ if:
  1. $\exists k : \; E_k(a) = 0$, $\mathrm{supp}(a) \subseteq A \in \Sigma_k$.
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Theorem

$h_1 = h_1^{\mathrm{at}}$, where h_1 is the subspace of L_1 of functions f such that

$$\|f\|_{h_1} := \left\| \left(\sum_k E_{k-1} |df_k|^2 \right)^{\frac{1}{2}} \right\|_1 < \infty$$

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- Question: Is there an atomic description of H_1?
- Answer: No, if we only consider classical atoms.

Atomic blocks

Definition

Let b be Σ-measurable. It is called a **(p)-atomic block** if:

1. $\exists k: \; E_k(b) = 0$, $\operatorname{supp}(b) \subseteq B \in \Sigma_k$.
2. $b = \sum_j \lambda_j a_j$, λ_j scalar.
3. Each **subatom** a_j satisfies the following properties:
   - $\exists k_j \geq k : \operatorname{supp}(a_j) \subseteq A_j \subseteq B$, $A_j \in \Sigma_{k_j}$.
   - $$\|a_j\|_p \leq \frac{1}{\mu(A_j)^{1/p'}} \frac{1}{k_j - k + 1}.$$

Set also

$$|b|_{H_{1,p}^{\mathrm{atb}}} = \inf_{b=\sum \lambda_j a_j} \sum_j |\lambda_j|.$$

We define

$$H_{1,p}^{\mathrm{atb}} = \left\{ f \in L_1(\Omega, \Sigma, \mu) : f = \sum_i b_i, \text{ each } b_i \text{ is a p-atomic block} \right\}.$$

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equipped with the norm

$$\|f\|_{H_{1,p}^{\mathrm{atb}}} := \inf_{\substack{f=\sum_i b_i,\\ b_i=\sum_j \lambda_{ij} a_{ij}}} \sum_i |b_i|_{H_{1,p}^{\mathrm{atb}}} = \inf_{\substack{f=\sum_i b_i,\\ b_i=\sum_j \lambda_{ij} a_{ij}}} \sum_{i,j} |\lambda_{ij}|.$$

Main result

Theorem (C., Parcet)

$$H_{1,p}^{\mathrm{atb}} = H_1$$

with equivalent norms.

Sketch of proof

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First inclusion: $\mathrm{BMO} \subseteq (H_{1,p}^{\mathrm{atb}})^*$. Given f, define $L_f(b) := E(bf)$.

$$\left| \int bf d\mu \right| = \left| \int b(f - E_k f) d\mu \right|$$

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$$\begin{aligned} \left| \int bf d\mu \right| &= \left| \int b(f - E_k f) d\mu \right| \\ &\leq \sum_j |\lambda_j| \left| \int a_j (f - E_k f) d\mu \right| \\ &=: \sum_j |\lambda_j| I_j. \end{aligned}$$

$$I_j \le \|a_j\|_p \left(\int_{A_j} |f - E_k f|^{p'} d\mu \right)^{\frac{1}{p'}}$$

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&\leq \left(\fint_{A_j} |f - E_{k_j} f|^{p'} d\mu \right)^{\frac{1}{p'}} + \frac{1}{k_j - k + 1} \sum_{l=k+1}^{k_j} \|E_l f - E_{l-1} f\|_\infty
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&\lesssim \|f\|_{\mathrm{bmo}^c} + \frac{k_j - k}{k_j - k + 1} \|f\|_{\mathrm{bmo}^d} \\
&\leq 2\|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

Second inclusion: $(H_{1,p}^{\text{atb}})^* \subseteq \text{BMO}$

Need of an additional tool:

Definition

Let $\Sigma_1 \subset \Sigma_2$ be σ-algebras, and X a Σ_2-measurable r.v. Y is a *conditional median* of X w.r.t Σ_1 if:

- Y is Σ_1-measurable.
- $\forall A \in \Sigma_1$,

$$\mu(A \cap \{X > Y\}) \leq \frac{1}{2}\mu(A) \geq \mu(A \cap \{X < Y\}).$$

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Remark: Y is a conditional median of X iff

$$E(|X - Y|) = \inf_{G\ \Sigma_1-\text{measurable}} E(|X - G|).$$

Theorem (Tomkins, '75)

Given a probability space (Ω, Σ, μ) and a r.v. X, there exists at least one conditional median Y of X w.r.t. any σ-algebra $\Sigma' \subset \Sigma$.

Given f, denote any conditional median of f w.r.t Σ_k by $\alpha_k f$.

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Definition

$$\|f\|_\circ := \max\left\{\sup_k \|E_k|f-\alpha_k f|^{p'}\|_\infty^{\frac{1}{p'}}, \sup_k \|\alpha_k f - \alpha_{k-1} f\|_\infty\right\}$$

The norm $\|\cdot\|_\circ$ does not depend on the election of conditional median.

It is easy to check:

$$\|f\|_{\mathrm{BMO}} \lesssim \|f\|_{\circ}.$$

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Now use properties of α_k to estimate

$$\|f\|_{\circ} \lesssim \|L_f\|_{H_{1,p}^{\mathrm{atb}}}.$$

$\square$

Motivation (too late!)

It is classical that

$$H_1(\mathbb{R}^n, dx) = \sum_{j=1}^{n+1} H_1^j(\mathbb{R}^n, dx).$$

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This is dual to

$$\mathrm{BMO}(\mathbb{R}^n, dx) = \bigcap_{j=1}^{n+1} \mathrm{BMO}^j(\mathbb{R}^n, dx).$$

(Garnett-Jones, Mei...).

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- Yes, if μ is doubling. Reason: H_1^j can be interpreted as a martingale space and it coincides with h_1^j.
- No, in general.

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where

$$\|f\|_{\mathrm{RBMO}^c} = \sup_{Q \text{ doubling cube}} \left(\fint_Q \left| f - \fint_Q f d\mu \right|^2 d\mu \right)^{\frac{1}{2}},$$

$$\|f\|_{\mathrm{RBMO}^d} = \sup_{Q \subset R \text{ and } Q, R \text{ doubling}} \left| \fint_Q f d\mu - \fint_R f d\mu \right|$$

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Remark: The result is true for measures which are not of polynomial growth.

Thank you!