Optimal Sobolev embeddings in mixed norm spaces

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Rearrangement-invariant Banach function spaces

Given a finite interval I, we denote $I^k = \overbrace{I \times \ldots \times I}^{k}$, $k \in \mathbb{N}$.

Definition

A mapping $\rho : \mathcal{M}_+(I^k) \to [0, \infty)$ is called a rearrangement-invariant Banach function norm (r.i. norm) if the following properties holds:

- $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(\alpha f) = |\alpha|\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$;
- $0 \leq g \leq f$ a.e. $\Rightarrow \rho(g) \leq \rho(f)$ (the lattice property);
- $0 \leq f_j \uparrow f$ a.e. $\Rightarrow \rho(f_j) \uparrow \rho(f)$ (the Fatou property);
- $\rho(\chi_{I^k}) < \infty$;
- $\displaystyle\int_{I^k} f(x)dx \leq C\rho(f)$, for some constant $C > 0$;
- $f^* = g^* \Rightarrow \rho(f) = \rho(g)$, where

$$f^*(t) = \inf\left\{s \geq 0 : \left|\left\{x \in I^k : |f(x)| > s\right\}\right| \leq t\right\}, \quad \text{for any } t \geq 0.$$

Rearrangement-invariant Banach function spaces

Definition

Given an r.i. norm ρ, the rearrangement-invariant Banach function space $X(I^k)$ (r.i. space) is the collection of all $f \in \mathcal{M}(I^k)$ for which $\rho(|f|) < \infty$. For each $f \in X(I^k)$, we define $\|f\|_{X(I^k)} = \rho(|f|)$.

Definition

Let $X(I^k)$ be an r.i. space and let $f \in X(I^k)$. We say that f has absolutely continuous norm (a.c. norm) if

$$\lim_{j\to\infty} \|f\chi_{E_j}\|_{X(I^k)} = 0,$$

for every sequence $\{E_j\}_{j\in\mathbb{N}}$ satisfying $\chi_{E_j} \to 0$ a.e. If every function in $X(I^k)$ has a.c. norm, then we say that $X(I^k)$ has a.c. norm.

Mixed norm spaces

Let $n \in \mathbb{N}$, $n \geq 2$ and $k \in \{1, \dots, n\}$. For any $x \in I^n$, we denote

$$\widehat{x_k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in I^{n-1}.$$

Definition

Let $X(I^{n-1})$ be an r.i. space. The Benedek-Panzone space $\mathcal{R}_k(X, L^\infty)$ is the collection of all $f \in \mathcal{M}(I^n)$ for which the quantity

$$\|f\|_{\mathcal{R}_k(X,L^\infty)} = \|\psi_k\|_{X(I^{n-1})}, \quad \psi_k(\widehat{x_k}) = \|f(\widehat{x_k}, \cdot)\|_{L^\infty(I)}$$

is finite.

Example

Let $n = 3$, $1 < p < \infty$ and let $f(x) = \chi_{A_1 \times A_2 \times A_3}(x)$. Then,

$$\psi_1(\widehat{x_1}) = \chi_{A_2 \times A_3}(\widehat{x_1}) \Rightarrow \|f\|_{\mathcal{R}_1(L^p,L^\infty)} = \|\psi_1\|_{L^p(I^{n-1})} = \big(|A_2||A_3|\big)^{1/p}.$$

$$\|f\|_{\mathcal{R}_2(L^p,L^\infty)} = \big(|A_1||A_3|\big)^{1/p}, \quad \|f\|_{\mathcal{R}_3(L^p,L^\infty)} = \big(|A_1||A_2|\big)^{1/p}$$

Mixed norm spaces

Definition

Let $X(I^{n-1})$ be an r.i. space. The mixed norm space $\mathcal{R}(X, L^\infty)$ is defined as follows

$$\mathcal{R}(X, L^\infty) = \bigcap_{k=1}^{n} \mathcal{R}_k(X, L^\infty).$$

For each $f \in \mathcal{R}(X, L^\infty)$, we set $\|f\|_{\mathcal{R}(X,L^\infty)} = \sum_{k=1}^{n} \|f\|_{\mathcal{R}_k(X,L^\infty)}$.

Example

- Let $n = 3$, $1 < p < \infty$ and let $f(x) = \chi_{A_1 \times A_2 \times A_3}(x)$. Then,

$$\begin{aligned}\|f\|_{\mathcal{R}(L^p,L^\infty)} &= \|f\|_{\mathcal{R}_1(L^p,L^\infty)} + \|f\|_{\mathcal{R}_2(L^p,L^\infty)} + \|f\|_{\mathcal{R}_3(L^p,L^\infty)} \\ &= |A_2|^{1/p}|A_3|^{1/p} + |A_1|^{1/p}|A_3|^{1/p} + |A_1|^{1/p}|A_2|^{1/p}.\end{aligned}$$

Sobolev spaces

We denote $\nabla u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$, where $\partial_{x_i} u$ is the distributional partial derivate of u with respect to the variable x_i.

Definition

Let $Z(I^n)$ be an r.i. space. We define the first-order Sobolev space $W^1 Z(I^n)$ as

$$W^1 Z(I^n) := \left\{ u \in L^1_{\mathrm{loc}}(I_n) : u \in Z(I^n) \text{ and } |\nabla u| \in Z(I_n) \right\},$$

with the norm $\|u\|_{W^1 Z(I^n)} = \|u\|_{Z(I^n)} + \||\nabla u|\|_{Z(I^n)}$.

By $W^1_0 Z(I^n)$, we will denote the closure of $C^\infty_c(I^n)$ in $W^1 Z(I^n)$.

Classical Sobolev embeddings

Theorem (Gagliardo-Nirenberg, Morrey and Sobolev)

- *If* $1 \leq p < n$, *then* $W_0^1 L^p(I^n) \hookrightarrow L^{pn/(n-p)}(I^n)$.
- *If* $p = n$, *then* $W_0^1 L^n(I^n) \hookrightarrow L^q(I^n)$, *for any* $n \leq q < \infty$.
- *If* $p > n$, *then* $W_0^1 L^p(I^n) \hookrightarrow L^\infty(I^n)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\sup_{x,y \in I^n} \frac{|u(x) - u(y)|}{|x - y|^{1-\frac{n}{p}}} \leq C \|u\|_{W_0^1 L^p(I^n)}, \quad u \in W_0^1 L^p(I^n).$$

Classical Sobolev embeddings

In 1938, S. L. Sobolev proved that

$$W^1L^p(I^n) \hookrightarrow L^{pn/(n-p)}(I^n), \quad \text{for } 1 < p < n;$$

his proof did not apply to $p = 1$. At the end of the 1950's, E. Gagliardo and L. Nirenberg found a method that worked in the case $p = 1$.

Theorem (E. Gagliardo, L. Nirenberg)

$W_0^1L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \hookrightarrow L^{n'}(I^n)$.

- E. Gagliardo, Ricerche Mat., **7** (1958), 102–137.
- L. Nirenberg, Annali della Scuola Normale Sup. Pisa, **13** (1959), 116–162.
- S. L. Sobolev, Mat. Sb., **46** (1938), 471–496.

Sobolev embeddings in r.i. spaces

R. Kerman and L. Pick studied the Sobolev embeddings among r.i. spaces. In particular, they solved the following problems:

- Given an r.i. range space $X(I^n)$, find the largest r.i. domain space, with a.c. norm, namely $Z(I^n)$, satisfying
$$W_0^1 Z(I^n) \hookrightarrow X(I^n).$$
This means that if $W_0^1 \widetilde{Z}(I^n) \hookrightarrow X(I^n) \Rightarrow \widetilde{Z}(I^n) \hookrightarrow Z(I^n)$.

- Given an r.i. domain space $Z(I^n)$, describe the smallest r.i. range space, namely $X(I^n)$, that verifies
$$W_0^1 Z(I^n) \hookrightarrow X(I^n).$$
That is, if $W_0^1 Z(I^n) \hookrightarrow \widetilde{X}(I^n) \Rightarrow X(I^n) \hookrightarrow \widetilde{X}(I^n)$.

- D.E. Edmunds; R. Kerman; L. Pick, J. Funct. Anal., **170** (2000), 307–355.
- R. Kerman; L. Pick, Forum Math., **18** (2006), 535–570.

Examples

Definition

Let $0 < p, q \leq \infty$. The Lorentz space $L^{p,q}(I^n)$ is the space of those $f \in \mathcal{M}(I^n)$ for which the quantity

$$\|f\|_{L^{p,q}(I^n)} = \left\| t^{1/p-1/q} f^*(t) \right\|_{L^q(I^n)}$$

is finite.

Theorem (R. Hunt, R. O'Neil, J. Peetre)

$W_0^1 L^1(I^n) \hookrightarrow L^{n',1}(I^n)$.

Theorem (R. Kerman and L. Pick)

The Lorentz space $L^{n',1}(I^n)$ is the smallest r.i. range space for which the Sobolev emebdding $W_0^1 L^1(I^n) \hookrightarrow L^{n',1}(I^n)$ holds.

Examples

Definition

Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. The Lorentz-Zygmund space $L^{p,q;\alpha}(I^n)$ is the space of all $f \in \mathcal{M}(I^n)$ such that the expression

$$\|f\|_{L^{p,q;\alpha}(I^n)} = \left\| t^{1/p-1/q} \big(\log\big(e\,|I|^n/t\big)\big)^{\alpha} f^*(t) \right\|_{L^q(0,|I|^n)}$$

is finite.

Theorem (V.G. Maz'ya, K. Hansson, H. Brézis and S. Wainger)

$W_0^1 L^n(I^n) \hookrightarrow L^{\infty,n;-1}(I^n)$.

Examples

Theorem (R. Kerman and L. Pick)

The space $L^{\infty,n;-1}(I^n)$ is the smallest r.i. range space satisfying

$$W_0^1 L^n(I^n) \hookrightarrow L^{\infty,n;-1}(I^n).$$

Theorem (R. Kerman and L. Pick)

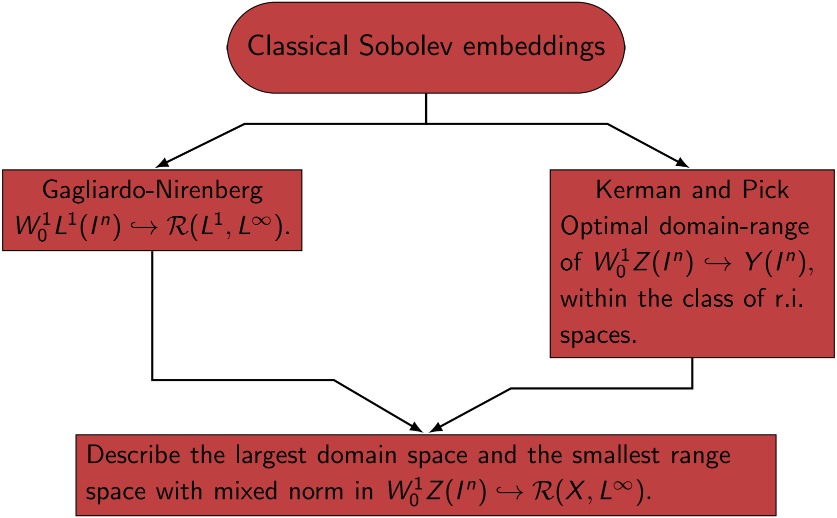
The space $Z_{L^{\infty,n;-1}}(I^n)$, with norm given by

$$\|f\|_{Z_{L^{\infty,n;-1}}(I^n)} = \left\| \int_{(\cdot)}^{|I|^n} s^{-1/n'} f^*(s)\,ds \right\|_{L^{\infty,n;-1}(I^n)}, \quad f \in \mathcal{M}(I^n)$$

is the largest r.i. domain space, with a.c. norm, for the Sobolev embedding

$$W_0^1 Z_{L^{\infty,n;-1}}(I^n) \hookrightarrow L^{\infty,n;-1}(I^n).$$

Motivation


Classical Sobolev embeddings
Gagliardo-Nirenberg
$W_0^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty)$.
Kerman and Pick
Optimal domain-range of $W_0^1 Z(I^n) \hookrightarrow Y(I^n)$, within the class of r.i. spaces.
Describe the largest domain space and the smallest range space with mixed norm in $W_0^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty)$.

Problem

Let $X(I^{n-1})$ be an r.i. space and let $Z(I^n)$ be an r.i space, with a.c. norm. Our aim is to study the Sobolev embedding

$$W_0^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty). \qquad (1)$$

We are interested in the following questions:

- Given a mixed norm space $\mathcal{R}(X, L^\infty)$, we want to find the largest r.i. domain space, with a.c. norm, satisfying (1).
- Let $Z(I^n)$ be an r.i. domain space, with a.c. norm. We would like to find the smallest range space of the form $\mathcal{R}(X, L^\infty)$ for which (1) holds.

Results

We reduce the Sobolev embedding to the boundedness of a Hardy type operator.

Proposition

Let $Z(I^n)$ be an r.i. space, with a.c. norm, and let $X(I^{n-1})$ be an r.i. space. Then, Sobolev embedding

$$W_0^1 Z(I^n) \hookrightarrow \mathcal{R}(X, L^\infty)$$

holds if and only if there exists a constant $C > 0$ such that

$$\left\| \int_{(\cdot)^{n'}}^{|I|^n} f(t) t^{-1/n'} dt \right\|_{\overline{X}(0,|I|^{n-1})} \leq C \, \|f\|_{\overline{Z}(0,|I|^n)}, \quad f \in \overline{Z}(0, |I|^n).$$

Results

We want to describe the optimal domain space for a given mixed norm space of the form $\mathcal{R}(X, L^\infty)$.

Theorem

Let $X(I^{n-1})$ be an r.i. space, with upper Boyd index $\overline{\alpha}_X < 1$. Then, the r.i. space $Z_{\mathcal{R}(X,L^\infty)}(I^n)$, with norm given by

$$\|f\|_{Z_{\mathcal{R}(X,L^\infty)}(I^n)} \approx \left\| \int_{(\cdot)^{n'}}^{|I|^n} f^*(t) t^{-1/n'} du \right\|_{\overline{X}(0,|I|^{n-1})}, \quad f \in \mathcal{M}(I^n)$$

satisfies

$$W_0^1 Z_{\mathcal{R}(X,L^\infty)}(I^n) \hookrightarrow \mathcal{R}(X, L^\infty).$$

Moreover, it is the largest r.i. domain space, with a.c. norm, for which this embedding holds.

Results

We want to describe the optimal range space of the form $\mathcal{R}(X, L^\infty)$ for a given domain space.

Theorem

Let $Z(I^n)$ be an r.i. space, with a.c. norm, and let $X_{W_0^1 Z, L^\infty}(I^{n-1})$ be the r.i. space whose associate space $X'_{W_0^1 Z, L^\infty}(I^{n-1})$ has norm

$$\|f\|_{X'_{W_0^1 Z, L^\infty}(I^{n-1})} = \left\| f^{**}((\cdot)^{1/n'}) \right\|_{\overline{Z}'(0,|I|^n)}, \quad f \in \mathcal{M}(I^{n-1}).$$

Then, the Sobolev embedding

$$W_0^1 Z(I^n) \hookrightarrow \mathcal{R}(X_{W_0^1 Z, L^\infty}, L^\infty)$$

holds and $\mathcal{R}(X_{W_0^1 Z, L^\infty}, L^\infty)$ is the smallest space of the form $\mathcal{R}(X, L^\infty)$ that verifies this embedding.

Examples

Theorem (E. Gagliardo, L. Nirenberg)

$W_0^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty)$.

Theorem

The mixed norm space $\mathcal{R}(L^1, L^\infty)$ *is the smallest range space of the form* $\mathcal{R}(X, L^\infty)$ *satisfying*

$$W_0^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty).$$

Theorem

The mixed norm space $\mathcal{R}(L^{\infty,n;-1}, L^\infty)$ *is the smallest space of the form* $\mathcal{R}(X, L^\infty)$ *satisfying*

$$W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Examples

Theorem

The space $Z_{\mathcal{R}(L^{\infty,n;-1},L^\infty)}(I^n)$, with norm given by

$$\|f\|_{Z_{\mathcal{R}(L^{\infty,n;-1},L^\infty)}(I^n)} \approx \left\| \int_{(\cdot)}^{|I|^n} f^*(s) s^{-1/n'} ds \right\|_{L^{\infty,n;-1}(I^n)}, \quad f \in \mathcal{M}(I^n)$$

is the largest r.i. domain space, with a.c. norm, satisfying

$$W_0^1 Z_{\mathcal{R}(L^{\infty,n;-1},L^\infty)}(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Domain space: $L^1(I^n)$

$L^{n',1}(I^n)$ is the smallest r.i. range space that verifies

$$W_0^1 L^1(I^n) \hookrightarrow L^{n',1}(I^n).$$

$\mathcal{R}(L^1, L^\infty)$ is the smallest range of the form $\mathcal{R}(X, L^\infty)$ satisfying $W_0^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty)$.

Problem: Are $L^{n',1}(I^n)$ and $\mathcal{R}(L^1, L^\infty)$ comparable?

Theorem (J. J. F. Fournier)

$$\mathcal{R}(L^1, L^\infty) \underset{\neq}{\hookrightarrow} L^{n',1}(I^n).$$

Corollary

$$W_0^1 L^1(I^n) \hookrightarrow \mathcal{R}(L^1, L^\infty) \underset{\neq}{\hookrightarrow} L^{n',1}(I^n).$$

- J. J. F. Fournier, Ann. Mat. Pura Appl. (4), **148** (1987), 51–76.

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$L^{\infty,n;-1}(I^n)$ is the smallest r.i. range space that verifies

$$W_0^1 L^n(I^n) \hookrightarrow L^{\infty,n;-1}(I^n).$$

$\mathcal{R}(L^{\infty,n;-1}, L^\infty)$, is the smallest range of the form $\mathcal{R}(X, L^\infty)$ for

$$W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty).$$

Problem: Are $L^{\infty,n,;-1}(I^n)$ and $\mathcal{R}(L^{\infty,n;-1}, L^\infty)$ comparable?

Proposition

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Corollary

$$W_0^1 L^n(I^n) \hookrightarrow \mathcal{R}(L^{\infty,n;-1}, L^\infty) \underset{\neq}{\hookrightarrow} L^{\infty,n;-1}(I^n).$$

Domain space: $L^n(I^n)$

$L^{\infty,n;-1}(I^n)$ is the smallest r.i. range space that verifies
$$W^1_0 L^n(I^n) \hookrightarrow L^{\infty,n;-1}(I^n).$$

$\mathcal{R}(L^{\infty,n;-1}, L^\infty)$, is the smallest range of the form $\mathcal{R}(X, L^\infty)$ for
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Domain space: $Z(I^n)$

Theorem

Let $X^{\mathrm{op}}(I^n)$ be the smallest r.i. range space satisfying

$$W_0^1 Z(I^n) \hookrightarrow X^{\mathrm{op}}(I^n),$$

and let $\mathcal{R}(X_{W_0^1 Z, L^\infty}, L^\infty)$ be the smallest space of the form $\mathcal{R}(X, L^\infty)$ for

$$W_0^1 Z(I^n) \hookrightarrow \mathcal{R}(X_{W_0^1 Z, L^\infty}, L^\infty).$$

Then, the following chain of embeddings holds

$$W_0^1 Z(I^n) \hookrightarrow \mathcal{R}(X_{W_0^1 Z, L^\infty}, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n).$$

Moreover, $X^{\mathrm{op}}(I^n)$ is the smallest r.i. space that verifies

$$\mathcal{R}(X_{W_0^1 Z, L^\infty}, L^\infty) \hookrightarrow X^{\mathrm{op}}(I^n).$$

The end

Thank You!!