Hardy inequalities for the self-adjointness of Dirac operators

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The free Dirac operator

$$H_0 = -i\alpha \cdot \nabla + m\beta, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3), \ \ \alpha_k, \beta \in \mathcal{M}_{4\times 4}(\mathbb{C}), \ \ m \geq 0$$

$$\beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \qquad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \qquad k = 1, 2, 3$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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H_0 is a matrix valued differential operator which acts on

$$\psi(x) : \mathbb{R}^3 \to \mathbb{C}^4, \quad \psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad \phi, \chi : \mathbb{R}^3 \to \mathbb{C}^2.$$

$$H_0\psi = \begin{pmatrix} m\mathbb{I}_2 & -i\sigma\cdot\nabla \\ -i\sigma\cdot\nabla & -m\mathbb{I}_2 \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

Purpose: To find $D \subset L^2(\mathbb{R}^3)^4$ and $V : D \to L^2(\mathbb{R}^3)^4$ s. t. $H = H_0 - V$ restricted to D is self-adjoint in $L^2(\mathbb{R}^3)^4$.

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Diagonal type potentials:

$$V(x) = \begin{pmatrix} w_1(x)\mathbb{I}_2 & 0 \\ 0 & w_2(x)\mathbb{I}_2 \end{pmatrix},$$

w_1 a real function or a singular radial and positive measure supported in $\mathbb{R}^3 \backslash \{0\}$ and $w_2 : \mathbb{R}^3 \to \mathbb{R}^+$.

- $w_1 < 0$: easy case.
- $w_1 \geq 0$: we need a Hardy inequality; for some $\lambda \in (-m, m)$

$$\int_{\mathbb{R}^3} w_1|\phi|^2 \leq \int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \phi|^2}{m + w_2 - \lambda} + (m + \lambda) \int_{\mathbb{R}^3} |\phi|^2.$$

The case $w_1 = w_2 = c/|x|$, $c \leq 1$ studied by Nenciu, Wüst and Esteban and Loss, among others.

Self-adjointness and extensions

Definition

A densely defined operator T on a Hilbert space is *symmetric* if and only if

$$(T\varphi, \psi) = (\varphi, T\psi) \quad \text{for all} \quad \varphi, \psi \in \mathcal{D}(T).$$

Definition

An operator T is *self-adjoint* if and only if T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.

H_0 and $H_0 - V$, V bounded with domain $H^1(\mathbb{R}^3)^4$ are self-adjoint.

The class of potentials

Definition

Let $\mathcal{A}$ be the class of potentials that contains all pairs of positive radial measurable functions, $V_1, V_2 : \mathbb{R}^3 \to \mathbb{R}^+$, that satisfy

$$A_+[V_1, V_2] := \sup_{r>0} \left[\frac{1}{r^2} \int_0^r (V_1(t) + V_2(t)) \; t^2 \; dt \right] < +\infty$$

and

$$A_-[V_1, V_2] := \sup_{r>0} \left[r^2 \int_r^\infty (V_1(t) + V_2(t)) \; \frac{dt}{t^2} \right] < +\infty.$$

Self-adjoint Extensions

Let $\mathcal{H}$ a Hilbert space with the following inner product

$$(\phi,\varphi)_{\mathcal{H}} := \int_{\mathbb{R}^3} (m - w_1 + \lambda)\phi \cdot \overline{\varphi} + \int_{\mathbb{R}^3} \frac{i\sigma \cdot \nabla\phi}{m + w_2 - \lambda} \cdot \overline{i\sigma \cdot \nabla\varphi}.$$

Define
$\mathcal{D} = \{(\phi,\chi) \in \mathcal{H} \times L^2(\mathbb{R}^3)^2 :$
$(m - w_1 + \lambda)\phi - i\sigma \cdot \nabla\chi \ , \ -i\sigma \cdot \nabla\phi + (-m - w_2 + \lambda)\chi \in L^2(\mathbb{R}^3)^2\}.$

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Theorem (A., 2011)

Let $V_1, V_2 \in \mathcal{A}$, $w_1, w_2 \geq 0$ s. t. $w_1(x) \leq c_1 V_1(|x|)$, $w_2(x) \leq c_2 V_2(|x|)$ and $c_1 c_2 \leq \frac{1}{\max\{A_+^2, A_-^2\}}$. Then H defined on $\mathcal{D}$ is self-adjoint. Furthermore, it is the unique s.a.e. of H on $\mathcal{C}_c^\infty(\mathbb{R}^3)^4$ s. t. $\mathcal{D} \subset \mathcal{H} \times L^2(\mathbb{R}^3)^2$.

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Sketch of Proof.

H is symmetric + $(H+\lambda)$ is a bijection from $\mathcal{D}$ to L^2 + Uniqueness

The Hilbert space $\mathcal{H}$:

$$(H+\lambda)\begin{pmatrix}\phi\\ \chi\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} m-w_1+\lambda & -i\sigma\cdot\nabla \\ -i\sigma\cdot\nabla & -m-w_2+\lambda \end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\Leftrightarrow \begin{cases} (m-w_1+\lambda)\phi - i\sigma\cdot\nabla\chi = 0 \\ -i\sigma\cdot\nabla\phi + (-m-w_2+\lambda)\chi = 0 \end{cases} \quad \Rightarrow \chi = \frac{-i\sigma\cdot\nabla\phi}{-m-w_2+\lambda}$$

$$\Rightarrow (m-w_1+\lambda)\phi - i\sigma\cdot\nabla\frac{-i\sigma\cdot\nabla\phi}{-m-w_2+\lambda} = 0.$$

Examples of potentials

- Let $0 \leq w_1(x) \leq \dfrac{c_1}{|x|}$ and $0 \leq w_2(x) \leq \dfrac{c_2}{|x|}$ for $c_1 c_2 \leq 1$.
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  (We gain freedom on the constants c_1, c_2.)

- Let $w_1(x) = a\delta_{|x|=R}$, $R, a > 0$, and $0 \leq w_2(x) \leq \frac{c}{|x|}$. If $ac \leq \dfrac{4}{9}$, the theorem for self-adjoint extensions holds.

Thank you!