The Pohozaev identity for the fractional Laplacian

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Abstract. The Pohozaev identity satisfied by solutions of $-\Delta u = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, is a classical result which has many applications. For instance, it leads to sharp nonexistence results, to uniqueness of solutions, monotonicity formulas, and to symmetry properties. Moreover, the identity is also used in other contexts such as the wave equation or harmonic maps.

The aim of this talk is to present the Pohozaev identity for the fractional Laplacian. In a joint work with Joaquim Serra, we establish a new identity satisfied for every bounded solution of the nonlocal semilinear Dirichlet problem $(-\Delta)^s u = f(x,u)$ in Ω, $u \equiv 0$ in $\mathbb{R}^n \setminus \Omega$. Here, $s \in (0,1)$, and $(-\Delta)^s$ stands for the fractional Laplacian in $\mathbb{R}^n$ —the infinitesimal generator of a symmetric and stable Lévy process. Surprisingly, from a nonlocal problem we obtain an identity with a boundary term (an integral over $\partial\Omega$) which is completely local.

To establish this identity we need to prove, among other things, that if u is a bounded solution then $u/\delta^s|_\Omega$ is C^α up to the boundary $\partial\Omega$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. In the fractional Pohozaev identity, the function $u/\delta^s|_{\partial\Omega}$ plays the role that $\partial u/\partial\nu$ plays in the classical one.